MATH30038 Advanced Group Theory Exam Solutions 2002

1. *For the purposes of this question, we define a* Dihedral Group *to be a group* $D = \langle x, y \rangle$ *where* x *and* y *are distinct involutions.*

   (a) *Classify all abelian dihedral groups.*
   Solution x and y are distinct elements of order 2. Let $z = xy$ which is clearly not an element of $\{1, x, y\}$. Now $G = \{1, x, y, z\}$ is closed under multiplication and $z^2 = 1$ so it is also closed under inversion. Thus G is a group of size 4, and has the following multiplicative structure: the product of any pair of different non-identity elements is the third non-identity element.

   (b) *Let* $z = xy$.

   (i) *Show that* $z^x = z^y = z^{-1}$ *and deduce that* $H = \langle z \rangle$ *is a normal subgroup of* D.
   Solution $z^x = x^{-1}xyx = yx = y^{-1}x^{-1} = z^{-1}$. Also $z^y = y^{-1}xyy = y^{-1}x^{-1} = z^{-1}$. Thus $H^x \leq H$ and $H^y \leq H$. Each of x and y is self-inverse so every $g \in G$ is a word in x and y, and therefore $H^g \leq H$ for every $g \in G$ and therefore $H \trianglelefteq D$.

   (ii) *Prove that* $xH = yH$, *and deduce that* G/H *is cyclic of order 1 or 2.*
   Solution $z = xy \in H$ so $xyH = H$ and therefore $xxyH = xH$ i.e. $yH = xH$. Every element of G/H is a word in xH and yH and is therefore a power of xH. However $x^2H = H$ and so $G/H = H \cup xH$, so $|G : H| = 2$ or 1 (as $xH \neq H$ or $xH = H$).

(iii) *Prove that $x \notin H$, and deduce that $G/H \simeq C_2$.*
Solution If $x \in H$ then x and z commute, so x and xz commute. Thus x and y commute and we are in the configuration dealt with in part (a), and in this case $x \notin H$ which is absurd. Thus $x \notin H$ so $xH \neq H$ and $|G : H| = 2$.

2. (a) *Exhibit a non-abelian group with non-trivial centre.*
Solution The dihedral group generated by two distinct involutions with product of order 4. We can exhibit this group as a subgroup of S_4: $G = \langle (1, 3), (1, 4)(2, 3) \rangle$.

(b) *Let $Z(G)$ denote the centre of the group G. Show that if $G/Z(G)$ is a cyclic group, then G must be abelian.*
Solution Suppose that $G/Z(G) = \langle xZ(G) \rangle$ so $G = \langle x, Z(G) \rangle$ is abelian.

(c) *Show that if G is a group such that Aut G is a cyclic group, then G must be abelian.*
Solution $G/Z(G) \simeq \mathrm{Inn}(G) \leq \mathrm{Aut}(G)$. A subgroup of a cyclic group is necessarily cyclic, so $G/Z(G)$ is cyclic and part (b) applies.

3. If A, B are groups, we let $\mathrm{Hom}(A, B)$ denote the set of homomorphisms with domain A and codomain B.

(a) *Explain what is meant be a* finitely generated *group.*
Solution G is finitely generated means that there is a finite set X such that $\langle X \rangle = G$. Here $\langle X \rangle$ simultaneously denotes the intersection of all subgroups of G wich contain X, and the set of all words on letters from X and their inverses. These two notions can be proved equivalent.

(b) *Suppose that G is a finitely generated group, and that H is a finite group. Show that Hom(G, H) is finite.*
Solution Suppose that G is generated by the finite set X. If $f : G \to H$ is a map, there are $|H|$ possible images for each $x \in X$. Given also that f is a homomorphism, the 'words' view on generation shows that the image of every $g \in G$ is determined by the images of the elements of X. Thus $\mathrm{Hom}(G, H)$ has size at most $|H|^{|X|}$ and so is finite.

(c) *Exhibit a group G and a finite group H such that $Hom(G, H)$ is an infinite set (and make it clear why $Hom(G, H)$ is infinite).*
Solution Let G be the set of infinite sequences (x_i) where each x_i is an element of H, a cyclic group of order 2. Composition in G is defined termwise. For each $j \in \mathbb{N}$ we have a group homomorphism ε_j defined by $(x_i) \mapsto x_j$. These maps are clearly group homomorphisms. Moreover they are diferent maps. This is because if $j \neq j'$ are natural numbers, let $y = (y_i) \in G$ where $y_i = 1$ if $i \neq j$ and $y_j \neq 1$. Now $(y)\varepsilon_j = y_j \neq 1 = (y)\varepsilon_{j'}$.

(d) Give an example of a subgroup H of a finitely generated group G where H is not finitely generated.
Solution Consider a subgroup G of $\mathrm{Sym}(\mathbb{Z})$. Let θ be the map 'add 2', and let ψ be the permutation $(1, 2)$. Let $G =\rangle\theta, \psi\rangle$. Thus G is finitely generated. Now let H be the group generated by all conjugates of ψ by (positive and negative) powers of θ. Each element of H has finite support, but every integer is moved by some element of H, so H cannot be finitely generated.

4. (a) *Suppose that G is a group with normal subgroups M, N such that $M \cap N = 1$. Prove that if $m \in M$ and $n \in N$, then $mn = nm$.*
Solution $m^{-1}n^{-1}mn = (n^{-1})^m \cdot m = m^{-1} \cdot m^n \in M \cap N = 1$. Thus $mn = nm$.

(b) *Suppose that p is a prime number, and that the group H has order p^2, then the group H must be abelian.* Let the conjugacy classes of G be $C_1, \dots, C_h$. Suppose that $x_i \in C_i$ for each i and let $c_i = |C_i|$ for every i. We may assume that $x_1 = 1$ and so $C_1 = \{1\}$. Every $C_i = |G : C_G(x_i)|$ and so must be $1, p$ or p^2. Now $p^2 = |G| = \sum_i c_i$ so the number of conjugacy classes of size 1 must be a multiple of p, and there is at least one such conjugacy class, so there must be at least p such classes. Let $\{x\}$ be such a class with $x \neq 1$. If $o(x) = p^2$ then G is cyclic and therefore abelian. Otherwise $o(x) = p$. Choose $y \in G - \langle x\rangle$. Now x is central so $H = \langle x, y\rangle$ has size at least $p + 1$ and is abelian. By Lagrange's theorem $H = G$ so G is abelian.

(c) *Prove that any group of order 35^2 must be abelian.* You may appeal to Sylow's theorems.
Solution The number of Sylow 5-subgroups must be 1 mod 5 and

divide 49 and so must be 1. The number of Sylow 7-subgroups must be 1 mod 7 and divide 25 and so must be 1. Thus there are unique normal subgroups P and Q of orders 25 and 49 respectively. Each of P and Q is abelian by part (b). Moreover elements of P commute with elements of Q by part (a). Now $|PQ| = |P| \cdot |Q|/|P \cap Q| = 35^2$. Thus $G = PQ$ so G must be abelian.