Conjugacy classes of the Alternating Group A_n

GCS: 17-xi-03

This note is to correct some insanity in Friday's lecture. In what follows, n is a positive integer and at least 2.

First we recall a general fact which we proved via working on a problem sheet:

Lemma Suppose that G is a group and that H and K are subgroups of G, then $|K : H \cap K| \leq |G : H|$.

Suppose that $x \in A_n \leq S_n$. Let the conjugacy class of x in S_n be C and the conjugacy class of x in A_n be D.

Theorem $C = D$ if and only x commutes with an odd element of S_n. If this is not the case, then $2|D| = |C|$.

Proof Apply the lemma with $S_n = G$, $A_n = H$ and $C_{S_n}(x) = K$. Note that $C_{A_n}(x) = A_n \cap C_{S_n}(x)$ so $|C_{S_n}(x) : C_{A_n}(x)| \leq |S_n : A_n| = 2$.

Now if this index is 1 if and only if $C_{S_n}(x) = C_{A_n}(x)$ which happens if and only if x commutes with no odd element of S_n. In these circumstances $|S_n : A_n| \cdot |A_n : C_{A_n}(x)| = |S_n : C_{A_n}(x)| = |S_n : C_{S_n}(x)|$ and so $2|D| = |C|$.

On the other hand if $|C_{S_n}(x) : C_{A_n}(x)| = 2$, then a factor of two can be cancelled from $|S_n : A_n| \cdot |A_n : C_{A_n}(x)| = |S_n : C_{A_n}(x)| = |S_n : C_{S_n}(x)| \cdot |C_{S_n}(x) : C_{A_n}|$ to yield that $|C| = |D|$, but $D \subseteq C$ so $C = D$. Conversely if $C = D$, then $|S_n : C_{S_n}(x)| = |A_n : C_{A_n}(x)|$. Now $|S_n : A_n| \cdot |A_n : C_{A_n}(x)| = |S_n : C_{A_n}(x)| = |S_n : C_{S_n}(x)| \cdot |C_{S_n}(x) : C_{A_n}|$ so $|S_n : A_n| = |C_{S_n}(x) : C_{A_n}| = 2$.

The proof is complete.

Example In S_3 the elements $(1,2,3)$ and $(1,3,2)$ are conjugate. In $A_3 = \langle(1,2,3)\rangle$ they are not (the group A_3 is abelian so different elements are never conjugate). The conjugacy class of $(1,2,3)$ is S_3 is $\{(1,2,3),(1,3,2)\}$ whereas is A_3 it is $\{(1,2,3)\}$. Notice that the centralizer of $(1,2,3)$ in S_3 is $\langle(1,2,3)\rangle$ so $(1,2,3)$ commutes with no odd elements of S_3.

Observations In the lecture I speculated that when a conjugacy class of S_n falls into two conjugacy classes in A_n, then the elements of one class might necessarily be the inverses of the elements of the other. Note that inverse elements do have the same cycle shape since you can invert an element in standard form by reversing each of its cycles. This speculation turns out to be false. In A_5 there are two conjugacy classes of elements with shape which is a 5-cycle. However $g = (2,5)(3,4) \in A_5$ and $g^{-1}(1,2,3,4,5)g = (1,5,4,3,2) = (1,2,3,4,5)^{-1}$.