Cauchy's Theorem

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An inductive approach to Cauchy's Theorem

CT for a finite abelian group A

Theorem Let A be a finite abelian group and suppose that p is a prime number which divides $|A|$. It follows that there is an element $g \in A$ with $o(g) = p$.

Proof If any proper subgroup has order divisible by p, then we can use an induction on $|A|$ to finish. Thus we may assume that every proper subgroup of A has order coprime to p. Let M be a proper subgroup of A of maximal size. Choose $x \in A$ with $x \notin M$. Let $X = \langle x \rangle$. Now $G = MX$ by maximality of M, so p divides

$$|G| = |M| \cdot |X|/|M \cap X| = |X| \cdot |M : M \cap X|.$$

Now $|M|$ and hence $|M : M \cap X|$ is coprime to p. Thus p divides $|X| = o(x) = ps$. Now $y = x^s \neq 1$ but $y^p = x^{o(x)} = 1$ and therefore y has order p.

CT for a finite group G

Theorem Let G be a finite group and suppose that p is a prime number which divides $|G|$. It follows that there is an element $g \in G$ with $o(g) = p$.

Proof We induct on $|G|$, the result being vacuously true for the trivial group. If any proper subgroup H of G has index $|G : H|$ coprime to p, then p divides $|H|$ and induction applies, so H contains an element of order p.

Thus we are done unless every proper subgroup of G has index divisible by p. In this case, if $\mathbf{C}$ is any conjugacy class of G which is not a singleton set, then choosing $y \in \mathbf{C}$, we find that

$$|\mathbf{C}| = |G : C_G(y)| \equiv 0 \bmod p.$$

Now $|G| \equiv 0 \bmod p$ and $|G|$ is the sum of the sizes of its conjugacy classes. Working modulo p we discover that the number of conjugacy classes of G which are singleton sets is a multiple of p. However, the set $\{t\}$ is a conjugacy class of G if and only if

$$t \in Z(G) = \{z \in G \mid zg = gz \forall g \in G\}.$$

Thus p divides $|Z(G)|$. Now apply Cauchy's theorem to the abelian group $Z(G)$ to produce an element of G of order p.