Group Theory: Math0038, Sheet 1

GCS

The course web site is available via `http://www.bath.ac.uk/~masgcs/`

1. Let G be a group. Suppose that $x \in G$. Let
$$C_G(x) = \{g \in G \mid gx = xg\} \subseteq G.$$
Prove that $C_G(x) \leq G$.

2. Let G be a group. Suppose that $S \subseteq G$. Let
$$C_G(S) = \{g \in G \mid gs = sg \forall s \in S\}.$$
Prove that $C_G(S) \leq G$.

3. Let G be a group. Suppose that $S \subseteq G$. Let
$$N_G(S) = \{g \in G \mid gS = Sg\}$$
where $gS = \{gs \mid s \in S\}$ and $Sg = \{sg \mid s \in S\}$. Prove that $C_G(S) \leq N_G(S) \leq G$.

4. Let G be a group. Suppose that $S \subseteq G$ and that $x \in N_G(S)$. Prove that $C_G(S)x = xC_G(S)$.

5. Let G be a finite group. Suppose that $\emptyset \neq H \subseteq G$ has the property that if $a, b \in H$, then $ab \in H$. Does it follow that $H \leq G$? What happens if we relax the condition that G is finite?

6. Suppose that G is a group and that $x \in G$. Prove that $(x^{-1})^{-1} = x$.

7. Does there exist a group G containing elements a, b such that $a^2 = b^2 = (ab)^3 = 1$?

8. Suppose that G is a group with the property that $x^2 = 1$ whenever x is an element of G. Show that G must be abelian.

9. (Challenge) Suppose that G is a group with the property that $x^3 = 1$ whenever x is an element of G. Show that G need not be abelian.