Group Theory: Math30038, Sheet 1

GCS: Solutions

The course web site is available via `http://www.bath.ac.uk/~masgcs/`

1. Let G be a group. Suppose that $x \in G$. Let
$$C_G(x) = \{g \in G \mid gx = xg\} \subseteq G.$$
Prove that $C_G(x) \leq G$.
Solution: $1 \in C_G(x) \neq \emptyset$. Moreover if $g \in G$ and $gx = xg$, then $xg^{-1} = g^{-1}x$ so if $a, b \in C_G(x)$, then $ab^{-1}x = axb^{-1} = xab^{-1}$ and therefore $ab^{-1} \in C_G(x)$.

2. *Let G be a group. Suppose that $S \subseteq G$. Let*
$$C_G(S) = \{g \in G \mid gs = sg \forall s \in S\}.$$
Prove that $C_G(S) \leq G$.
Solution: We have
$$C_G(S) = \cap_{s \in S} C_G(s)$$
and since the intersection of subgroups is a subgroup, we are done.

3. Let G be a group. Suppose that $S \subseteq G$. Let
$$N_G(S) = \{g \in G \mid gS = Sg\}$$
where $gS = \{gs \mid s \in S\}$ and $Sg = \{sg \mid s \in S\}$. Prove that $C_G(S) \leq N_G(S) \leq G$.
Solution: The fact that $N_G(S)$ is a group follows the outline of the proof that $C_G(x)$ is a group. The condition to be in $C_G(S)$ is stronger than that to be in $N_G(S)$ so $C_G(S) \leq N_G(S) \leq G$.

4. *Let G be a group. Suppose that $S \subseteq G$ and that $x \in N_G(S)$. Prove that $C_G(S)x = xC_G(S)$.*
Solution: We will show that $C_G(S) = x^{-1}C_G(S)x$ which will suffice. Suppose that $s \in S$, then $xs = s'x$ for some $s' \in S$. Now shoose any $c \in C_G(S)$, then
$$x^{-1}cxs = x^{-1}cs'x = x^{-1}s'cx = x^{-1}(xsx^{-1})cx = sx^{-1}cx$$
and therefore $sx^{-1}c \in C_G(S)$ and so $x^{-1}C_G(S)x \leq C_G(S)$. Replacing x by x^{-1} in this argument yields that $xC_G(S)x^{-1} \leq C_G(S)$. Premultiplying by x^{-1} and postmultiplying by x gives $C_G(S) \leq x^{-1}C_G(S)x$. Now we have two mutually reverse inclusions so $C_G(S) = x^{-1}C_G(S)x$ for all $x \in N_G(S)$.

5. *Let G be a finite group. Suppose that $\emptyset \neq H \subseteq G$ has the property that if $a, b \in H$, then $ab \in H$. Does it follow that $H \leq G$? What happens if we relax the condition that G is finite?*
Solution: If $h \in H$, then $\langle h \rangle \leq H$ and so $o(h)$ is a natural number n. If $h = 1$ then $h^{-1} = h \in H$. Otherwise $n > 1$ and then $h^{-1} = h^{n-1} \in H$ so H is a subgroup of G. In the event that G is infinite, things fall apart. For example perhaps $G = \mathbb{Z}$ under addition, the $\mathbb{N}$ is a non-empty additively closed subset which is not a subgroup since $-1 \notin \mathbb{N}$.

6. Suppose that G is a group and that $x \in G$. Prove that $(x^{-1})^{-1} = x$.
Solution: $x^{-1}x = 1 = x^{-1}(x^{-1})^{-1}$. Premultiplying by x gives the result.

7. *Does there exist a group G containing elements a, b such that $a^2 = b^2 = (ab)^3 = 1$?*
Solution: Yes, the trivial group will do it. More interestingly, let G be the group S_3, and put $a = (1, 2)$ and $b = (2, 3)$. Now the orders of a, b and $ab$are exactly "as advertized".

8. *Suppose that G is a group with the property that $x^2 = 1$ whenever x is an element of G. Show that G must be abelian.*
Solution: Suppose that $a, b \in G$, then $abab = 1 = baab$. Postmultiplying by ba yields $ab = ba$.

9. *(Challenge) Suppose that G is a group with the property that $x^3 = 1$ whenever x is an element of G. Show that G need not be abelian.*
Solution: Consider the set of 3 by 3 upper triangular matrices with entries in $F = \mathbb{Z}_3$, the field of integers modulo 3 which have 1s on the leading diagonal. It is easy to verify that these 27 matrices form a group. Each matrix is of the form $I + \Delta$ wheer I is the 3 by 3 identity matrix and Δ is strictly upper triangular. Note that Δ^3 is the zero matrix, so the inverse of $I + \Delta$ is $I - \Delta + \Delta^2$ and moreover $(I + \Delta)^3 = I + 3\Delta + 3\Delta^2 = I$ so ev ery element of the group has order dividing 3. The matrices $I + E_{12}$ and $I + E_{23}$ do not commute as may be verified by direct calculation; here E_{ab} is the 3 by 3 matrix with entries in F where the entrie in the i-th row and j-th column is $\delta_{ia}\delta_{jb}$ where δ is the Kronecker delta.