Group Theory: Math30038, Sheet 2

GCS

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1. Let G be a finite group of even order. Show that G must contain an element of order 2.
   Solution: The number of elements which are distinct from their own inverse is even (such elements occur in pairs). Therefore the number of elements which co-incide with their own inverses is even, However, 1 co-incides with its own inverse, so there are an odd number of non-identity elements which co-incide with their own inverses, i.e. there are an odd number of elements of order 2, so there is at least one of them.

2. Suppose that G is a cyclic group and that $|G| = 12$. Make a list of all the subgroups of G, and draw a "Hasse diagram". This is a diagram of dots and lines (as shown in a lecture), where the inclusion $A \leq B$ is indicated by a line or sequence of lines going up from the dot labelled A to the dot labelled B. You can label the lines with corresponding indices. Stare at your diagram, and think about the following set: $\{d \in \mathbb{N} \mid d \text{ divides } 12\}$.

3. Suppose that G is a group and that $|G| = 4$. By considering the orders of the elements of G (or otherwise), prove that G must be abelian.
   Solution: If every element of G sqaures to 1, then we have shown that G is abelian. Thus we are done unless G contains an element of order not dividing 2 (but dividing 4 by Lagrange's theorem). Thus we are left with the case that G is a cyclic group. However, cyclic groups are abelian so we are done.

4. Suppose that G is a group and that $|G| = 6$.
   (a) Show that if G is abelian, then G must be cyclic.
   (b) Show that if G is nonabelian, then G must contain an element x of order 2 and an element y of order 3, and moreover it must be the case that $xyx = y^{-1}$.

5. Consider a cube in Euclidean space. The group R of all rigid motions of the cube (reflections not allowed) has order 24. Paint the vertices black and white so that if two vertices are joined by an edge, then they have opposite colour. Let G be the subgroup of R consisting of those rigid motions which preserve the colouring (black corners go to black corners

and white corners to white ones). Describe the 12 elements of G. Show that G has no subgroup of order 6.

Solution: This group G has order dividing 24 by Lagrange's theorem, and is not R because rotation through $\pi/2$ about an axis joining the centre of a face to the centre of the opposite face is not in G. Rotations through π about these three axes are in G, as are the eight rotations through $\pm 2\pi/3$ about great diagonals, and the identity element. Thus the order of G is at least 12 and is therefore exactly 12. Suppose (for contradiction) that G had a subgroup of H order 6. Suppose that $x \in G$, then if $Hx = H$ then $x \in H$ so $x^2 \in H$. Conversely if $Hx \neq H$, then $Hx^2 \neq Hx$ so $Hx^2 = H$ and therefore $x^2 \in H$. Thus all squares of elements of G are in H. However, the eight elements of order 3 in G are all squares (of their squares) so $8 \leq 6$. This is the required contradiction.

6. Let $\Omega = \mathbb{Z}$. Let $\mathrm{Sym}(\Omega)$ denote the group of all bijections from Ω to Ω under composition of maps. Let $\alpha \in \mathrm{Sym}(\Omega)$ be such that $(x)\alpha = x + 2 \ \forall x \in \mathbb{Z}$ and let β be the bijection which swaps 0 and 1, but fixes all other integers. Let $G = \langle \alpha, \beta \rangle$, so G is a finitely generated subgroup of $\mathrm{Sym}(\Omega)$. Let $X = \{\alpha^{-i}\beta\alpha^i \mid i \in \mathbb{Z}\}$. Let $H = \langle X \rangle$. Show that H is not finitely generated (i.e. there does not exist a finite subset Y of H such that $H = \langle Y \rangle$). *Hint: the* support *of a bijection is the set of elements which are not fixed by the bijection. Show that all elements of H have finite support, and think about the consequences of this.*

7. Consider the group of bijections from $\mathbb{R}^3$ to $\mathbb{R}^3$ under composition of maps. Let i denote the bijection which is clockwise rotation through $\pi/2$ about the x axis, and j denote the bijection which is clockwise rotation through $\pi/2$ about the y axis. Let $k = ij$. Describe k geometrically. Discuss the group $\langle i, j \rangle$: what is its order and what does its multiplication table look like?