Group Theory: Math30038, Sheet 3

GCS

The course web site is available via `http://www.bath.ac.uk/~masgcs/`

1. Let ϕ denote the Euler ϕ-function. Prove that for every integer n we have
$$\sum \phi(d) = n$$
where the sum is taken over all natural numbers d which divide n.

2. Suppose that G is a finite abelian group. Suppose that p is a prime number which divides $|G|$. Prove that there is an element $g \in G$ such that $o(g) = p$. *Hint: multiply together all the cyclic subgroups of* G.

3. Suppose that G is a group and that K, L are both normal subgroups with the property that $K \cap L = 1$ (i.e. K and L intersect to form the trivial subgroup consisting of the identity element). Prove that every element of K commutes with every element of L. *Hint: consider elements of the form* $k^{-1}l^{-1}kl$ *where* $k \in K$ *and* $l \in L$.

4. Suppose that G is a group and that H is a subgroup of G of finite index. Suppose that K is also a subgroup of G. Prove that $|K : H \cap K| \leq |G : H|$. What can you say if this inequality is an equality?

5. Let G be a group. Suppose that $H \leq G$ and that $|G : H| = 2$. Prove that $H \trianglelefteq G$. Can one derive the same conclusion when 2 is replaced by 3?