Group Theory: Math30038, Sheet 3

GCS

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1. Let ϕ denote the Euler ϕ-function. Prove that for every integer n we have
$$\sum \phi(d) = n$$
where the sum is taken over all natural numbers d which divide n.
Solution: In a cyclic group of order n, there are exactly $\phi(d)$ elements of order d where d is any divisor of n.

2. Suppose that G is a finite abelian group. Suppose that p is a prime number which divides $|G|$. Prove that there is an element $g \in G$ such that $o(g) = p$. *Hint: multiply together all the cyclic subgroups of G.*
Solution: Follow the hint. If each element of G has order coprime to p, then it follows from the formula for the size of a product of subgroups that G has size coprime with p. This is not the case so there is an element $x \in G$ of order pm. Now $y = x^m$ has order p.

3. Suppose that G is a group and that K, L are both normal subgroups with the property that $K \cap L = 1$ (i.e. K and L intersect to form the trivial subgroup consisting of the identity element). Prove that every element of K commutes with every element of L. *Hint: consider elements of the form $k^{-1}l^{-1}kl$ where $k \in K$ and $l \in L$.*
Solution: $k^{-1}L^k = L$ and $l^{-1}Kl = K$ by normality so $k^{-1}l^{-1}k \in L$ and $l^{-1}kl \in K$. Therefore $k^{-1}l^{-1}kl \in K \cap L = \{1\}$. Thus $kl = lk$ wherever $k \in K$ and $l \in L$.

4. Suppose that G is a group and that H is a subgroup of G of finite index. Suppose that K is also a subgroup of G. Prove that $|K : H \cap K| \leq |G : H|$. What can you say if this inequality is an equality?
Solution: Let T be a right transversal for $H \cap K$ in K, so if $t, t' \in T$

are distinct, then $t, t' \in K$ but $tt'^{-1} \notin H \cap K$. Therefore $tt'^{-1} \notin H$. It follows that Ht, Ht' are distinct cosets. The inequality is established. We claim that equality holds if and only if $HK = G$. We prove this as follows. If $HK = G$ then it is possible to choose a right transversal S for H in G consisting of elements of K. Now if s, s' are distinct elements of S then $ss'^{-1} \notin H$ and so $ss'^{-1} \notin H \cap K$. bananas.

5. Let G be a group. Suppose that $H \leq G$ and that $|G : H| = 2$. Prove that $H \trianglelefteq G$. Can one derive the same conclusion when 2 is replaced by 3?
   Solution: Suppose that $g \in G$. If $g \in H$, then $gH = H = Hg$. On the other hand if $g \notin G$, then $gH \neq H$, but there are only two left cosets of H in G so $gH = G - H$ (i.e. the set of elemenst of G which are not elements of H). Similarly $Hg = G - H$ so $gH = Hg$. Thus H is a normal subgroup of G. The argument will not work when 2 is replaced by 3, for we may let $G = S_3$, and put $H = \langle (1, 2) \rangle$. Now H has order 2 and therefore index 3 in G. Moreover $(2, 3)H = \{(2, 3), (1, 2, 3)\}$ but $H(2, 3) = \{(2, 3), (1, 3, 2)\}$.