Group Theory: Math30038, Sheet 4

GCS

1. Suppose that G acts on a set Ω. If $\alpha \in \Omega$, we let
$$G_\alpha = \{g \in G \mid \alpha g = \alpha\}.$$
Now suppose that $\beta, \gamma \in \Omega$ are such that $\beta h = \gamma$ for some $h \in G$. Show that $G_\gamma = h^{-1} G_\beta h$.

2. Let P be a group of order p^n where p is a prime number. Suppose that P acts on a finite set Q of size q where p does not divide q. Show that this action of P has a fixed point (i.e. there is $\alpha \in Q$ such that $\alpha g = \alpha \forall g \in P$).

3. In how many essentially different ways can one colour the edges of a regular octahedron using c colours (where each edge is monochromatic, and two colourings are deemed the same if one can moved to the other by a rigid motion – and reflections are not allowed).

4. Let G be a group with subgroups H and K, each of finite index in G. Prove that $H \cap K$ has finite index in G.

5. Let G be a group and suppose that $H \leq G$ and $|G : H| < \infty$. By considering the groups $g^{-1}Hg$ as g ranges over G (or otherwise), prove that G has a normal subgroup N with $|G : N| < \infty$ and $N \leq H \leq G$.

6. Let G be a group and suppose that $x, y \in G$. Prove that $o(xy) = o(yx)$.