Group Theory: Math30038, Sheet 4

GCS

1. *Suppose that G acts on a set Ω. If $\alpha \in \Omega$, we let*

$$G_\alpha = \{g \in G \mid \alpha g = \alpha\}.$$

Now suppose that $\beta, \gamma \in \Omega$ are such that $\beta h = \gamma$ for some $h \in G$. Show that $G_\gamma = h^{-1} G_\beta h$.
Solution: Suppose that $x \in G_\beta$, then $\gamma h^{-1} x h = \beta h \cdot h^{-1} x h = \beta x h = \beta h = \gamma$ so $h^{-1} G_\beta h \subseteq G_\gamma$. Now $\beta = \gamma h^{-1}$ so a similar argument shows that $h G_\gamma h^{-1} \subseteq G_\beta$. Premultiplying by h^{-1} and postmultiplying by h it follows that $G_\gamma \subseteq h^{-1} G_\beta h$. We have an inclusion and its reverse, so $G_\gamma = h^{-1} G_\beta h$.

2. *Let P be a group of order p^n where p is a prime number. Suppose that P acts on a finite set Q of size q where p does not divide q. Show that this action of P has a fixed point (i.e. there is $\alpha \in Q$ such that $\alpha g = \alpha \forall g \in P$).*
Solution: If $\xi \in Q$, then the stabilizer (isotropy group) of ξ is denoted P_ξ, and the orbit of ξ has size $|P : P_\xi|$ which is a power of p. Now count Q by adding up the sizes of the orbits of P acting on Q to discover that at least one orbit must have size 1, else q would be divisible by p.

3. *In how many essentially different ways can one colour the edges of a regular octahedron using c colours (where each edge is monochromatic, and two colourings are deemed the same if one can moved to the other by a rigid motion – and reflections are not allowed).*
Solution: A regular octahedron has 8 identical equilateral faces, and its group of rotational symmetries G has order 24. There are 4 axes of symmetry through the centres of faces and the centre of the

opposite face, giving rise to 8 rotations A of order 3. There are 3 axes of symmetry through a vertex and the opposite vertex, giving rise to 6 rotations B of order 4 and 3 rotations C of order 2. There are 6 axes of symmetry through the centre of an edge and the centre of the opposite edge, giving rise to 6 rotations D of order 2, and there is the identity map E. We have described $8 + 6 + 3 + 6 + 1 = 24$ all of which are different, and so have a list of all the elements of G. Now let Ω be the set of coloured octahedra, a set of size c^{12}. We seek to count the orbits of G acting on Ω by using the counting principle not due to Burnside.

Element type	Number of this type	|Fix|
A	8	c^4
B	6	c^2
C	3	c^4
D	6	c^4
E	1	c^{12}

The number of essentially different edge colourings of the octahedron is therefore

$$\frac{c^{12} + 17c^4 + 6c^2}{24}.$$

For example when $c = 2$ this is 183.

4. Let G be a group with subgroups H and K, each of finite index in G. Prove that $H \cap K$ has finite index in G.
Solution It follows from Sheet 3, Question 4, that $|K : H \cap K| \leq |G : H| < \infty$. Now $|G : H \cap K| = |G : K| \cdot |K : K \cap H| < \infty$.

5. *Let G be a group and suppose that $H \leq G$ and $|G : H| < \infty$. By considering the groups $g^{-1}Hg$ as g ranges over G (or otherwise), prove that G has a normal subgroup N with $|G : N| < \infty$ and $N \leq H \leq G$.*
Solution: The solution to the previous question shows that the intersection of two subgroups of finite index in G is of finite index in G. A straightforward induction yields that the intersection of finitely many subgroups of finite index in G is of finite index in G. Now, it is a routine matter to check (please do it) that each set $g^{-1}Hg$ is a subgroup of G (where g is an arbitrary element of G). Let $\widehat{H} = \cap_{g \in G} g^{-1}Hg \leq G$. If $x \in G$ and $y \in \widehat{H}$, then $x^{-1}yx \in (gx)^{-1}Hgx$ for every $g \in G$ so $x^{-1}yx \in g^{-1}Hg$ for every $g \in G$. Thus $x^{-1}\widehat{H}x \leq \widehat{H}$. Similarly

$x\widehat{H}x^{-1} \le \widehat{H}$ and therefore $\widehat{H} \le x^{-1}\widehat{H}x$ for every $x \in G$ so $\widehat{H}$ is a normal subgroup of G.

If T is a right transversal for H in G, then it is easy to verify that $g^{-1}T$ is a right transversal for $g^{-1}Hg$ in G (please do it). Therefore each group $g^{-1}Hg$ is of finite index in G. We will be finished if we can show that there are only finitely many groups $g^{-1}Hg$ as g ranges over G. Suppose that $a, b \in G$ and that $a^{-1}Ha \neq b^{-1}Hb$ so $(ab^{-1})^{-1}Hab^{-1} \neq H$ it follows that $ab^{-1} \notin H$ and so a and b are in different right cosets of H in G. There are only finitely many right cosets of H in G so there are only finitely many groups $g^{-1}Hg$ as g ranges over G.

6. *Let G be a group and suppose that $x, y \in G$. Prove that $o(xy) = o(yx)$.*
   Solution: $y(xy)^n = (yx)^n y$ so $(xy)^n = 1$ if and only if $(yx)^n = 1$. Thus the orders of xy and yx co-incide.