Group Theory: Math30038, Sheet 5

GCS

1. Let $G = S_n$ be the symmetric group on $\{1, 2, \ldots, n\}$, so $|G| = n!$ and the elements of G are the permutations of $\{1, 2, \ldots, n\}$.

   (a) Suppose that $(a_1, a_2, \ldots, a_t) \in G$ is a cycle, and that $g \in G$. Show that $g^{-1}(a_1, a_2, \ldots, a_t)g = (a_1 g, a_2 g, \ldots, a_t g)$.

   (b) Each element of G can be expressed as a product of disjoint cycles (elements of G are *disjoint* if their supports are disjoint). Show that the number of conjugacy classes in S_n is the number of ways of writing n as an ascending sum $a_1 + a_2 + \cdots + a_t$ of positive integers $a_1 \leq a_2 \leq \cdots \leq a_t$. *Thus there are 3 conjugacy classes in* S_3 *because 3 can be written as an ascending sum in three ways:* $3, 1+2, 1+1+1$. *Also in* S_4 *there are 5 conjugacy classes because 4 is* $4, 1+3, 1+1+2, 2+2$ *and* $1+1+1+1$.

   (c) Determine the number of conjugacy classes in S_5, and the size of each conjugacy class, and describe the centralizer in G of a chosen representative of each conjugacy class.

2. Let $G = S_n$. Let $x = (1, 2, \ldots, n) \in G$. Prove that $C_G(x) = \langle x \rangle$.

3. Show that in S_4 there is a non-identity element y such that $C_G(y) \neq \langle y \rangle$.

4. Suppose that G is a finite group. Show that the number of elements in each conjugacy class of G must divide G.

5. Let G be a group with a subgroup H such that $g^{-1}Hg \subseteq H$ for every $g \in G$. Prove that $g^{-1}Hg = H$ for every $g \in G$.

6. (Challenge) Does there exist a group G containing an element g and a subgroup H such that $g^{-1}Hg \subseteq H$ but $g^{-1}Hg \neq H$.

7. Suppose that G is a group and that $H \leq G$. Choose $g \in G$. Prove that $g^{-1}Hg \leq G$.

8. Let G be a finite group of order n which has t conjugacy classes. Elements x and y are each selected uniformly at random from G. What is the probability that x and y commute? Does this make sense for abelian groups?

9. Show that a finite group with exactly two conjugacy classes must have two elements.

10. Let G be a containing H a subgroup of finite index. Let $S = \{x^{-1}Hx \mid x \in G\}$. Let G act on S by conjugation so if $K \in S$ then $K \cdot g = g^{-1}Kg$. Verify that this is a group action, and deduce that $|S| = |G : N_G(H)|$ where $N_G(H) = \{g \in G \mid gH = Hg\}$. Deduce that $|S|$ is finite and divides $|G : H|$.