Group Theory: Math30038, Sheet 5

Solutions GCS

1. *Let* $G = S_n$ *be the symmetric group on* $\{1, 2, \ldots, n\}$*, so* $|G| = n!$ *and the elements of* G *are the permutations of* $\{1, 2, \ldots, n\}$*.*

   (a) *Suppose that* $(a_1, a_2, \ldots, a_t) \in G$ *is a cycle, and that* $g \in G$*. Show that* $g^{-1}(a_1, a_2, \ldots, a_t)g = (a_1g, a_2g, \ldots, a_tg)$*.*
   Solution: If $x \neq a_ig$ for any i, then $xg^{-1} \neq a_i$ for any i, so $xg^{-1}(a_1, a_2, \ldots, a_t)g = xg^{-1}g = x$. However, if $x = a_ig$ for some i, then $xg^{-1}(a_1, a_2, \ldots, a_t)g = a_igg^{-1}(a_1, a_2, \ldots, a_t)g$ $= a_i(a_1, a_2, \ldots, a_t)g = a_{i+1}g$. Thus we can describe $g^{-1}(a_1, a_2, \ldots, a_t)g$ by the notation $(a_1g, a_2g, \ldots, a_tg)$.

   (b) *Each element of* G *can be expressed as a product of disjoint cycles (elements of* G *are* disjoint *if their supports are disjoint). Show that the number of conjugacy classes in* S_n *is the number of ways of writing* n *as an ascending sum* $a_1 + a_2 + \cdots + a_t$ *of positive integers* $a_1 \leq a_2 \leq \cdots \leq a_t$*.* Thus there are 3 conjugacy classes in S_3 because 3 can be written as an ascending sum in three ways: $3, 1+2, 1+1+1$. Also in S_4 there are 5 conjugacy classes be cause 4 is $4, 1+3, 1+1+2, 2+2$ and $1+1+1+1$.
   Solution: Conjugation sends cycles to cycles (of the same length). Morover, the calculation performed for Question 1 shows how g must be chosen to effect the required conjugation. In particular any two cycles of length r are conjugate in S_n by a suitably chosen g. Conjugation sends disjoint cycles to conjugate disjoint cycles so conjugacy classes in S_n consist of all elements with a given 'cycle shape'. Thus the cycle shapes characterize the conjugacy classes, and these shapes can be specified by listing the lengths of the disjoint cycles in ascending order.

(c) *Determine the number of conjugacy classes in S_5, and the size of each conjugacy class, and describe the centralizer in G of a chosen representative of each conjugacy class.*
Solution: Here is a transversal for the conjugacy classes: id, $(1,2)$, $(1,2,3)$, $(1,2,3,4)$, $(1,2,3,4,5)$, $(1,2)(3,4,5)$ and $(1,2)(3,4)$. The corresponding conjugacy classes have sizes $1, 10, 20, 30, 24, 20$ and 15 and happily these numbers sum to $120 = 5!$. The corresponding centralizers must have orders $120, 12, 6, 4, 5, 6$ and 8. This enables us to identify them as $G = S_5$, $\langle (1,2),\ (3,4,5),\ (3,4)\rangle$, $\langle (1,2,3),\ (4,5)\rangle$, $\langle (1,2,3,4)\rangle$, $\langle (1,2,3,4,5)\rangle$, $\langle (1,2),(3,4,5)\rangle$ and $\langle (1,2,3,4),(1,2)\rangle$.

2. *Let $G = S_n$. Let $x = (1,2,\ldots,n) \in G$. Prove that $C_G(x) = \langle x\rangle$.*
Solution: The conjugacy class of x has size $(n-1)!$ so $C_G(x)$ has index $(n-1)!$ and therefore order n. However $\langle x\rangle \leq C_G(x)$ and $|\langle x\rangle| = n$ so $C_G(x) = \langle x\rangle$.

3. *Show that in S_4 there is a non-identity element y such that $C_G(y) \neq \langle y\rangle$.*
Solution: (1,2) centralizes (1,2)(3,4) (and vice versa).

4. *Suppose that G is a finite group. Show that the number of elements in each conjugacy class of G must divide G.*
Solution: Suppose that $x \in G$ and that C is the conjugacy class of x. We know that $|C| = |G : C_G(x)|$ but $|G| = |G : C_G(x)| \cdot |C_G(x)|$ and we are done.

5. *Let G be a group with a subgroup H such that $g^{-1}Hg \subseteq H$ for every $g \in G$. Prove that $g^{-1}Hg = H$ for every $g \in G$.*
Solution: This is a standard trick. If $g^{-1}Hg \subseteq H$ for every $g \in G$, then replacing g by g^{-1} we obtain that $gHg^{-1} \subseteq H$ for every $g \in G$. Therefore $H \subseteq g^{-1}Hg$ for every $g \in G$ and so $H = g^{-1}Hg$ for every $g \in G$.

6. *(Challenge) Does there exist a group G containing an element g and a subgroup H such that $g^{-1}Hg \subseteq H$ but $g^{-1}Hg \neq H$.*
Solution: Yes. Let G be the set of 2 by 2 invertible rational matrices. Let H be the set of upper unitriangular matrices with integer entry in the top right position. Let g be the diagonal matrix $diag(1,2)$.

7. *Suppose that G is a group and that $H \leq G$. Choose $g \in G$. Prove that $g^{-1}Hg \leq G$.*
Solution: $g^{-1}1g \in g^{-1}Hg \neq \emptyset$. If $g^{-1}ag, g^{-1}bg \in g^{-1}Hg$, then $g^{-1}agg^{-1}bg = g^{-1}abg \in g^{-1}Hg$, As for inversion, the inverse of $g^{-1}ag$ is $g^{-1}a^{-1}g \in g^{-1}Hg$,

8. Let G be a finite group of order n which has t conjugacy classes. Elements x and y are each selected uniformly at random from G. What is the probability that x and y commute? Does this make sense for abelian group s?
Solution: Let $\Gamma = \{(x,y) | x, y \in G, xy = yx\}$. Now the required probability is
$$p = \frac{1}{|G|^2}|\Gamma| = \frac{1}{|G|^2}\sum_{x \in G} |C_G(x)|$$
and by not Burnside's counting lemma this is t/n.

9. Show that a finite group with exactly two conjugacy classes must have two elements.
Solution: We have proved that the number of elements in a conjugacy class of a finite group must divide the order of the group. If the group order is n this forces $n-1$ to divide n which forces $n = 2$.

10. Let G be a containing H a subgroup of finite index. Let $S = \{x^{-1}Hx \mid x \in G\}$. Let G act on S by conjugation so if $K \in S$ then $K \cdot g = g^{-1}Kg$. Verify that this is a group action, and deduce that $|S| = |G : N_G(H)|$ where $N_G(H) = \{g \in G \mid gH = Hg\}$. Deduce that $|S|$ is finite and divides $|G : H|$.
Solution: Suppose that $A \in S$ so that $A = x^{-1}Hx$ for some $x \in G$. Take elements $a, b \in G$. Now $A \cdot 1 = 1^{-1}A1 = A$ and $A \cdot ab = b^{-1}a^{-1}Aab = (A \cdot a) \cdot b$. Thus we have a group action. There is a single orbit of this action, and the orbit is in bijective correspondence with the right cosets of the stabilizer of H. This is $\{g \in G \mid g^{-1}Hg = H\}$ a group known as $N_G(H)$ and called the normalizer of H in G (and of course $H \leq N_G(H)$ and in fact $H \trianglelefteq N_G(H)$). Thus $|S| = |G : N_G(H)|$ but of course $|G : H| = |G : N_G(H)| \cdot |N_G(H) : H|$ and we were given that $|G : H| < \infty$. Thus $|S|$ is finite and a divisor of $|G : H|$.