Group Theory: Math30038, Sheet 6

GCS

1. Consider the group D of rigid symmetries of a regular n-gon (which may be turned over). Prove that this group has order $2n$, is non-abelian, and can be generated by two elements each of order 2. Show that D has a cyclic subgroup of index 2.

2. Consider the group D of rigid symmetries of the integers: so D is the group of all bijections θ from $\mathbb{Z}$ to $\mathbb{Z}$ which preserve distance. Thus θ must have the property that if $x, y \in \mathbb{Z}$, then $|x - y| = |(x)\theta - (y)\theta|$. Prove that this group has infinite order, is non-abelian, and can be generated by two elements each of order 2. Show that D has a cyclic subgroup of index 2.

3. Let $D = \langle x, y\rangle$ where $o(x) = o(y) = 2$ and $x \neq y$. Let $z = xy$ and put $H = \langle z\rangle$.

   (a) Prove that $x^{-1}zx = y^{-1}zy = z^{-1}$.

   (b) Prove that $x, y \notin H$.

   (c) Prove that $|G : H| = 2$.

   (d) Let $n = o(z) \in \mathbb{N} \cup \{\infty\}$. For each possible value of n let G be called D_n. Show that the multiplication in D_n is completely determined (i.e. the number n nails down the group).

   (e) For each $n \in \mathbb{N} \cup \{\infty\}$, determine the centre of D_n.

   (f) Determine the conjugacy classes of D_8.

   (g) Do you recognize D_6?

4. Suppose that G is a non-abelian finite group of order $2p$ where p is a prime number. Prove that G is generated by two elements order 2.

5. We define a subgroup Q of S_8 by letting $i = (1,2,3,4)(5,6,7,8)$, $j = (1,5,3,7)(2,8,4,6)$ (**and NOT** $(2,6,4,8)$ **as earlier stated**) and put $Q = \langle i, j \rangle$. Let $k = ij$ and $z = i^2$. *This group was the basis of William Rowan Hamilton's generalization of the complex numbers called the* Quaternions.

   (a) Show that $i^2 = j^2 = k^2 = z$ and $z^2 = 1$.

   (b) Show that $ij = k$, $jk = i$ and $ki = j$.

   (c) Show that $ji = zk$, $kj = zi$ and $ik = zj$.

   (d) Show that z is in the centre of Q.

   (e) Show that $Q = \langle i \rangle \cup z\langle i \rangle$.

   (f) Show that $|Q| = 8$.

   (g) Show that Q and D_8 (in Question 1) are both non-abelian groups of order 8, but they contain different numbers of elements of order 4.

   (h) Determine the conjugacy classes of Q.

   (i) On which bridge are the quaternions inscribed?

6. Let G denote the set of invertible n by n matrices with complex entries. This is a group under multiplication of matrices. Give a transversal for the conjugacy classes of G. *Hint: the course MA20012 does this (and not much else).*

7. Show that if N is a normal subgroup of G, then N must be a union of conjugacy classes of G. Deduce that the only normal subgroups of A_5 are 1 and A_5, but that A_4 has a normal subgroup M which is neither 1 nor A_4.