Group Theory: Math30038, Sheet 6

Solutions GCS

1. *Consider the group D of rigid symmetries of a regular n-gon (which may be turned over). Prove that this group has order $2n$, is non-abelian, and can be generated by two elements each of order 2. Show that D has a cyclic subgroup of index 2.*
   Solution Each axis of symmetry of a regular n-gon either passes through a vertex and the midpoint of the opposite side (if n is odd) or a pair of opposite vertices or mid-points of sides (if n is even), Either way, these give rise to n elements of order 2 in D which are relections in these axes. Label the vertices with the numbers from 1 to n in clockwise order. Suppose that $n = 2r + 1$ is odd. In this event let $x = (2, n)(3, n-1) \cdots (r+1, r+2)$ and $y = (1, 2)(n, 3) \cdots (r+1, r+3)$. Now $z = xy = (1, 2, 3, \ldots, n)$ and $yx = z^{-1} \neq z$ since $n \geq 3$ for non-triviality. Thus D is non-abelian. In the degenerate case $n = 2$ the group is in fact abelian. Now $\langle z \rangle$ is the cyclic group of order n consisting of rotations, and $x\langle z \rangle$ is the set of n reflections mentioned earlier. This exhausts D and $D = \langle x, y \rangle$. Next suppose that $n = 2r$ is even. This time let

   $$x = (1, 2)(n, 3) \cdots (r+2, r+3)$$

   and

   $$y = (1, 3)(n, 4) \cdots (r+1, r+3)$$

   so $xy = (1, 2, 3, \ldots, n)$ and the same story unfolds.

2. *Consider the group D of rigid symmetries of the integers: so D is the group of all bijections θ from $\mathbb{Z}$ to $\mathbb{Z}$ which preserve distance. Thus θ must have the property that if $x, y \in \mathbb{Z}$, then $|x - y| = |(x)\theta - (y)\theta|$. Prove that this group has infinite order, is non-abelian, and can be*

generated by two elements each of order 2. Show that D has a cyclic subgroup of index 2.
Solution Let

$$x = (1, -1)(2, -2)(3, -3)\cdots$$

and

$$y = (0, 1)(-1, 2)(-2, 3)\cdots.$$

Now xy is the map which sends z to $z+1$ for every integer z. Now $\langle xy\rangle$ is a cyclic subgroup of D. Any element t of D which is not in $\langle xy\rangle$ must send 0 to c and 1 to $c-1$. Now $t(xy)^{-c+1}$ sends 0 to 1 and 1 to 0. The definition of D ensures that $t(xy)^{-c+1} = y$ and so $t = y(xy)^{c-1} \in y\langle xy\rangle$. Thus the index of $\langle xy\rangle$ in D is at most 2. However, $y \notin \langle xy\rangle$ since y reverses the direction of the integers. Thus $|D : \langle xy\rangle| = 2$.

3. *Let $D = \langle x, y\rangle$ where $o(x) = o(y) = 2$ and $x \neq y$. Let $z = xy$ and put $H = \langle z\rangle$.*

   (a) *Prove that $x^{-1}zx = y^{-1}zy = z^{-1}$.*
   Solution $x^{-1}zx = x^{-1}xyx = yx = y^{-1}x^{-1} = z^{-1}$. Also $y^{-1}zy = y^{-1}xy^2 = y^{-1}x^{-1} = z^{-1}$.

   (b) *Prove that $x, y \notin H$.*
   Solution If $x \in H$, then $x^{-1}zx = z^{-1}$ but also $x^{-1}zx = z$ since H is cyclic and therefore abelian. Thus $z^2 = 1$. A cyclic group can contain at most one subgroup of order 2, and so $x = z = xy$ so $y = 1$ which contradicts $o(y) = 2$. Therefore $x \notin H$. If $y \in H$ then $x = zy \in H$ which we know is not the case, so $y \notin H$.

   (c) *Prove that $|G : H| = 2$.*
   Solution Since $x^{-1} = x$ and $y^{-1} = y$, and both x and y square to 1, any element of D must be either 1, or $xyx\cdots xyx$ or $yxy\cdots yxy$ or a power of z. Now $xyx\cdots xyx \in x\langle z\rangle$ and $yxy\cdots yxy = xxyxy\cdots yxy \in x\langle z\rangle$. Thus the index of H is at most 2. Since $x \notin H$ it follows that $|G : H| = 2$.

   (d) *Let $n = o(z) \in \mathbb{N}\cup\{\infty\}$. For each possible value of n let G be called D_n. Show that the multiplication in D_n is completely determined (i.e. the number n nails down the group).*
   Solution We have shown that in each group D there is a cyclic subgroup $H = \langle z\rangle$ of index 2, and that $xH \neq H$. Therefore every element of D_n is uniquely expressible at $x^{\varepsilon}h$ where $\varepsilon \in \{0, 1\}$

and $h \in H$. Note that $z^x = z^{-1}$ so $h^x = h^{-1}$ for all $h \in H$. Multiplication is as follows: suppose that $h, k \in H = \langle z \rangle$. We have

$$\begin{aligned} h \cdot k &= hk \\ h \cdot xkk &= xh^x k = xh^{-1}k \\ xh \cdot k &= xhk \\ xh \cdot xk &= h^x k = h^{-1}k \end{aligned}$$

Thus the only issue is how multiplication happens in H, but $H = \langle s \rangle$ is cyclic and its multiplication is determined by the single number $n = o(s)$.

(e) *For each $n \in \mathbb{N} \cup \{\infty\}$, determine the centre of D_n.*
Solution There is no group D_1. The group D_2 is of order 4 and so is abelian. Let us suppose that $n \geq 3$. Now $\langle xy \rangle$ contains at most one element of order 2. If n is odd or infinite, there is no such element, but if $n = 2r$ is even, there is such an element $w = z^r$ and since $w^x = w^y = w$, w is central. If $t = xz^m$ is in D_n but not in $H = \langle xy \rangle$ for some integer m, then $t^y = yxyyz^m y = xxyxyyz^m y = xz^2 z^{-m}$. Now $t^y = 1$ if and only if $z^{2m} = z^2$ so $o(z)$ divides $2m - 2$. However $t^x = t$ if and only if $xz^m = xz^{-m}$ and so $o(z)$ divides $2m$. Thus if t is central then $o(z)$ must divide both $2m - 2$ and $2m$ and so must divide 2. Thus for $n > 2$ the centre of D_n is the trivial subgroup unless n is finite and even, in which case the centre of D_n is $\{1, z^{n/2}\}$, a cyclic subgroup of order 2.

(f) *Determine the conjugacy classes of D_8.*
Solution I meant to ask about D_4, so we will do both cases. Let G be the group under discussion. It is easy to verify that the centralizer of x has order 4, and is $\{1, z^2, x, xz^2\}$. Therefore the conjugacy class of x in D_4 has size 2. Note that $x^y = yxy = xxyxy = xz^2$. Thus the conjugacy class of x is $\{x, xz^2\}$. Now $y = xz$ and the centralizer of xz at least contains $1, xz, z^2$ and xz^3 so the conjugacy class of y has size at most 2. However $y^x = xyx = zx = xz^{-1} = xz^3$. Thus the conjugacy class of y $(= xz)$ is $\{xz, xz^3\}$. The centre of G is $\{1, z^2\}$ and so each of the elements of this set is in a conjugacy class of size 1. Finally, the elements z and z^3 are non-central and therefore are in conjugacy classes of size greater than 1. The remaining conjugacy class is therefore

$\{z, z^3\}$. In summary, the conjugacy classes of D_4 are

$$\{1\},\ \{z^2\},\ \{z, z^3\},\ \{x, xz^2\}$$

and

$$\{xz, xz^3\}.$$

Let us work with D_8. The centralizer of x contains $1, x, z^4$ and xz^4 so the conjugacy class of x has size at most $16/4 = 4$. Now $x^y = yxy = xz^2$, and $x^{yx} = xz^6$ and $x^{yxy} = xz^2z^2 = xz^4$. Thus the conjugacy class of x is $\{x, xz^2, xz^4, xz^6\}$. Similar calculations reveal that the conjugacy class of $y = xz$ is $\{xz, xz^3, xz^5, xz^7\}$. The elements of H are all centralized by H and so are either central, or are in conjugacy classes of size 2. Each element of H is conjugate to its inverse. Therefore the conjugacy classes of D_8 are

$$\{x, xz^2, xz^4, xz^6\},\ \{xz, xz^3, xz^5, xz^7\},\ \{1\},\ \{z^4\},\ \{z, z^7\},\ \{z^2, z^6\}$$

and

$$\{z^3, z^5\}.$$

(g) *Do you recognize D_6?*
Solution I meant to ask about D_3, a non-abelian group of order 6, which is a copy of S_3. Notice that S_3 is generated by two distinct elements of order 2 (as one should expect).

4. Suppose that G is a non-abelian finite group of order $2p$ where p is a prime number. Prove that G is generated by two elements order 2.
Solution The fact that G is non-abelian forces the prime p to be odd. The group G contains an element x of order 2 and an element t of order p by Cauchy's theorem. Now $G = \langle x, t\rangle$ is non-abelian so x and t do not commute. Now $y = x^t \neq 1$ and $y^2 = t^{-1}xxt = 1$ so y has order 2. Let $T = \langle x, y\rangle$ so T has even order more than 2, and dividing $2p$. Therefore $T = G$ and by previous theory, G is a copy of D_p.

5. We define a subgroup Q of S_8 by letting $i = (1, 2, 3, 4)(5, 6, 7, 8)$, $j = (1, 5, 3, 7)(2, 8, 4, 6)$ (**and NOT** $(2, 6, 4, 8)$ **as earlier stated**) and put $Q = \langle i, j\rangle$. Let $k = ij$ and $z = i^2$. *This group was the basis of William Rowan Hamilton's generalization of the complex numbers called the* Quaternions.

(a) Show that $i^2 = j^2 = k^2 = z$ and $z^2 = 1$.
Solution This is routine.

(b) Show that $ij = k$, $jk = i$ and $ki = j$.
Solution This is routine.

(c) Show that $ji = zk$, $kj = zi$ and $ik = zj$.
Solution This is routine.

(d) Show that z is in the centre of Q.
Solution $z = i^2 = j^2$ so z commutes with both i and j, so z is central.

(e) Show that $Q = \langle i \rangle \cup z\langle i \rangle$.
Solution This is nonsense. $z \in \langle i \rangle$ and therefore $\langle i \rangle \cup z\langle i \rangle = \langle i \rangle$. However, for any element $u \notin \langle i \rangle$ it will be the case that $Q = \langle i \rangle \cup u\langle i \rangle$. The choices for u are j, zj, k and zk. Let us choose u to be k, and demonstrate this decomposition in this case. Any element of Q is a word in i and j. Now $j = ik^{-1} = ik^3$ so every element of Q is a word in i and k (with positive exponents). Now any occurrence of ik can be replaced by kiz Using this repeatedly the ks can migrate to the left. When this is done use $z = i^2$ to eliminate z. If the power of k involved is even, then use $k^{2n} = i^{2n}$ to eliminate k. If the power of k involved is odd, all but one k can be eliminated. Thus $Q = \langle i \rangle \cup k\langle i \rangle$.

(f) Show that $|Q| = 8$.
Solution This follows immediately from the previous part.

(g) Show that Q and D_8 (in Question 1) are both non-abelian groups of order 8, but they contain different numbers of elements of order 4. .
Solution I meant to write 'Show that Q and D_4 (in Question 1) are both non-abelian groups of order 8,' I fear that D_8 has order 16. In D_4 there are 5 involutions; there are 4 reflections and one rotation through π. In Q however, only z has order 2. One may verify tha t the remaining six non-identity elements of Q all have order 4. Now Q is non-abelian since $ij \neq ji$. D_4 is non-abelian because $xy \neq yx$.

(h) *Determine the conjugacy classes of Q.*
Solution They are

$$\{1\},\ \{z\},\ \{i, zi\},\ \{j, zj\} \text{ and } \{k, zk\}.$$

There are any number of ways to demonstrate this.

(i) *On which bridge are the quaternions inscribed?*
Solution The Sir William Rowan Hamilton Bridge, Dublin.

6. Let G denote the set of invertible n by n matrices with complex entries. This is a group under multiplication of matrices. Give a transversal for the conjugacy classes of G. *Hint: the course MA20012 does this (and not much else).*
Solution Jordan normal forms with "λ" non-zero.

7. Show that if N is a normal subgroup of G, then N must be a union of conjugacy classes of G including the conjugacy class of the identity element. Deduce that the only normal subgroups of A_5 are 1 and A_5, but that A_4 has a normal subgroup M which is neither 1 nor A_4.
Solution Since $g^{-1}Ng = N$ for all $g \in G$, it follows that if $n \in B$, then the conjugacy class of n is a subset of N. The conjugacy classes of A_5 are the conjugates of id, $(1, 2, 3)$, $(1, 2)(3, 4)$, $(1, 2, 3, 4, 5)$ and $(1, 2, 3, 5, 4)$. These conjugacy classes have sizes $1, 20, 15, 12$ and 12 respectively. The only sums these sizes which include 1 and which divide 60 are 1 and 60, so A_5 is a simple group. An analogous calculation for A_4 yields that the conjugacy classes of A_4 have sizes $1, 3, 4$ and 4. In this case $1 + 3$ is a divisor of 12, so it is possible that there is a normal subgroup of size 4. It so happens that

$$\{id,\ (1, 2)(3, 4),\ (1, 3)(2, 4),\ (1, 4)(2, 3)\}$$

is a (normal) subgroup of A_4 of this order.