Group Theory: Math30038, Sheet 7

GCS

1. Suppose that $n \geq 3$. Let $g \in G = S_n$ be an n-cycle. Recall that all n-cycles are conjugate in S_n.
   (a) Prove that $C_G(g) = \langle g \rangle$.
   (b) Deduce that $Z(G) = 1$.
   (c) Show that $\mathrm{Aut}(G)$ has a subgroup which is isomorphic to G.

2. Suppose that G is a group and that $\mathrm{Aut}(G)$ is the trivial group.
   (a) Show that G must be abelian.
   (b) By considering the inversion map, deduce that $g^2 = 1$ for all $g \in G$.
   (c) Write G additively, so that G becomes a vector space over the field $\mathbb{Z}_2$ of "integers modulo 2". You may assume that every vector space has a basis. Deduce that G must either be cyclic of order 2 or the trivial group.

3. Suppose that G is a group and that $\mathrm{Aut}(G)$ is finite. Prove that $|G : Z(G)|$ must be finite.

4. (Challenge) Suppose that G is a group and that $\mathrm{Aut}(G)$ is cyclic. Prove that G must be abelian.

5. Prove that for a finite group, the notions of isomorphism, monomorphism and epimorphism all co-incide.

6. In each case give an example of a group G and a map $\zeta : G \to G$ which satisfies the specified property.

(a) An epimorphism (i.e. a surjective homomorphism) but not an isomorphism (suggestion for G: fix a prime number p and consider those complex numbers z such that $z^{p^n} = 1$ for some natural number n which may vary as z varies).

(b) A monomorphism (i.e. an injective homomorphism) but not an isomorphism (suggestion for G: $\mathbb{Z}$ under addition).

7. Suppose that G is a finitely generated group (i.e. there is a finite subset X of G such that $\langle X \rangle = G$) and that every conjugacy class of G is finite. Prove that $|G : Z(G)|$ is finite.

8. Let V be a vector space of dimension n over a field F which has q elements. Let $\mathrm{GL}(V)$ denote the group of linear isomorphisms from V to V (a group under composition of maps). Determine $|\mathrm{GL}(V)|$.

9. (a) Suppose that G is a group acting on a set Ω. Define a map $\eta : G \longrightarrow \mathrm{Sym}(\Omega)$ by $(g)\eta : \omega \mapsto \omega \cdot g$ for all $\omega \in \Omega$ and for all $g \in G$. Prove that η is a homomorphism of groups.

   (b) Suppose that $\eta : G \longrightarrow \mathrm{Sym}(\Omega)$ is a group homomorphism. Show that one may define a group action of G on the set Ω by defining $\omega \cdot g$ to be $(\omega)((g)\eta)$ for all $\omega \in \Omega$ and for all $g \in G$. Show that this is indeed a group action.

   (c) Show that the procedures outlined in parts (a) and (b) are mutually inverse (i.e. if you perform one, then the other, then you recover the situation in which you started).