Group Theory: Math30038, Sheet 8

GCS Solutions

1. Let P be a Sylow p-subgroup of the finite group G. Suppose that $N \trianglelefteq G$.

   (a) *Show that* $P \cap N \in Syl_p(N)$.
   Solution $P \cap N \leq P$ so $|P \cap N|$ is a power of p. Also $PN/N \simeq P/P \cap N$ so $|PN : N| = |P : P \cap N|$ but

   $$|PN : N| \cdot |N : P \cap N| = |PN : P \cap N| = |PN : P| \cdot |P : P \cap N|$$

   and therefore $|N : P \cap N| = |PN : P|$. Now $|PN : N|$ is a divisor of $|G : P|$ and so is coprime to p. Thus $P \cap N$ is a Sylow p-subgroup of N.

   (b) *Show that* $PN/N \in Syl_p(G/N)$.
   Solution PN/N has p-power order by the 2nd isomorphism theorem. Now $|G : PN|$ divides $|G : P|$ an so is coprime to p. Now $|G/N : PN/N| = |G : PN|$ since if T is a left transveral for PN in G, then $\{tN \mid t \in T\}$ is easily seen to be a left transversal for PN/N in G/N.

2. *Show that every group of order* 15 *must be abelian.*
   Solution Sylow's theorem tells us that there are unique Sylow 5-subgroups and 3-subgroups P and Q respectively. These subgroups must be normal in G since conjugation must leave each one invariant. Now for all $a \in P$ and $b \in Q$ we have $a^{-1}b^{-1}ab \in P \cap Q = 1$ so a and b commute. Now if $a \in A$, then $A \leq C_G(a)$ since A is cyclic (of prime order). We have just shown that $B \leq C_G(A)$ so $\langle A, B\rangle \leq C_G(a)$. Now $\langle A, B\rangle$ has order divisible by both 3 and 5 and so $G = \langle A, B\rangle$. Thus $a \in Z(G)$ so $A \leq Z(G)$. Similarly $B \leq Z(G)$ so $G = \langle A, B\rangle \leq Z(G)$ and therefore G is abelian.

3. Show that every group of order 35 must be abelian.
Solution The solution is a copy of the previous solution.

4. *Show that there is no non-abelian finite simple group of order less than 60.*
Solution A group of p-power order has a non-trivial centre. This will prevent G from being simple unless $G = Z(G)$ is abelian (in which case G will be simple if and only if $|G| = p$ but this is not needed). Thus there is no non-abelian finite simple group of prime power order. Now using the results of the next question we may focus on groups of the following orders: 24, 40, 48, 54, 30 and 56.

   (a) A simple group of order 24 would have 3 Sylow 2-subgroups, and so there would (see Poincaré's theorem) be a non-trivial homomorphism from G to S_3 which will be injective by simplicity and therefore 24 would divide 6 which is absurd.

   (b) A group G of order 40 must have a unique Sylow 5-subgroup and so can not be simple.

   (c) A simple group of order 48 must have 3 sylow 2-subgroups, and so (see —G— = 24) we can deduce that 48 divides 6 which is absurd.

   (d) A group of order 54 must have a Sylow 3-subgroup H of index 2 in G. Now any subgroup of index 2 is normal so G can not be simple.

   (e) A simple group of order 30 would have 6 Sylow 5-subgroups. Since any pair of these groups intersects in the identity, it follows that there are 24 elements of order 5. Similarly this this group must have 10 Sylow 3-subgroups and so 20 elements of order 3. However, $24 + 20 > 30$ so this is absurd.

   (f) A simple group of order 56 would have 8 Sylow 7-subgroups, any pair of which would intersect in the trivial group. Thus there would be 48 elements of order 7. This leaves 8 elements remaining, which must all belong to a Sylow 2-subgroup. This Sylow 2-subgroup is unique and therefore invariant under conjugation and thus is normal in G. This is absurd.

5. Let p and q be distinct prime numbers.

(a) *Show that a group of order pq can not be simple.*

Solution Suppose w.l.o.g. that $p < q$. By Sylow's theorem there is a unique Sylow q-subgroup Q. Now Q is invariant under conjugation by elements of G, and so must be a normal subgroup of G.

(b) *Show that a group of order p^2q can not be simple.*

Solution Suppose (for contradiction) that G is simple. By Sylow's theorem the number of Sylow q-subgroups is p or p^2, and this number must be congruent to 1 modulo q. Therefore q divides $p-1$ or $p^2-1=(p-1)(p+1)$. In the first event $q<p$. In the second event $q<p$ or $(p,q)=(2,3)$.

Now, if $(p,q)=(2,3)$ it follows that $|G|=12$. In a non-simple group of order 12 there must be 4 Sylow 3-subgroups, any pair of which intersect in the identity. There are therefore 8 elements of order 3. The number of elements of order dividing 4 is therefore at most $12-8=4$. Let P be a Sylow 2 subgroup of G, a group of size 4, all elements of P will have order dividing 4. Thus P must consist of all the elements of G which are not of order 3. Thus there is a unique Sylow 2-subgroup which violates simplicity.

Thus we may assume that $q<p$. Now (see Poincaré's theorem) we have a non-trivial homomorphism $G \to S_q$ which must be injective by the simplicity of G. Now by Lagrange's theorem $|G|$ divides $q!$ and so p divides $q!$ which is false.

(c) *Show that a group of order p^2q^2 can not be simple.*

Solution Assume (for contradiction) that G is simple. The number of Sylow q subgroups q and must be p or p^2 (by simplicity) so q divides $p-1$ or divides $p^2-1=(p-1)(p+1)$. Thus either $q<p$ or $q=3, q=2$ (so $|G|=36$). Reversing the roles of p and q we obtain that either $p<q$ or $|G|=36$. Thus we are done unless $|G|=36$, but then a Sylow 3-subgroup has index 4 and so (see Poincaré's theorem) there is a non-trivial homomorphism from G to S_4 which must be injective by simplicity. Now by Lagrange's theorem 36 divides 24 which is absurd.