Algebra I: Math0008, Sheet 9

GCS

There next tutorials for this course are on Monday November 29th and Tuesday November 30th. The next lecture will be on Wednesday December 1st.

The course web site is available via `http://www.bath.ac.uk/~masgcs/`

1. Let V be an inner product space over $\mathbb{C}$.
   (a) Suppose that $\mathbf{u}, \mathbf{v} \in V$. Show that $(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{u}) \in \mathbb{R}$.
   (b) Suppose that $\mathbf{u}, \mathbf{v} \in V$. Show that $(\mathbf{u}, \mathbf{v}) - (\mathbf{v}, \mathbf{u})$ is purely imaginary.
   (c) Suppose that $\mathbf{u}, \mathbf{v} \in V$ and that $\lambda \in \mathbb{C}$. Show that $(\lambda\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \overline{\lambda}\mathbf{v})$.

2. Let $V = \mathbb{R}^4$. For each $\mathbf{u}, \mathbf{v} \in V$ where
   $$\mathbf{u} = (u_1, u_2, u_3, u_4),\ \ \mathbf{v} = (v_1, v_2, v_3, v_4)$$
   we put
   $$(\mathbf{u}, \mathbf{v}) = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4.$$
   Show that V is an inner product space.

3. Let W be the subspace of V spanned by $\mathbf{x}_1 = (1, 0, 0, 0)$, $\mathbf{x}_2 = (0, 1, 0, 0)$, and $\mathbf{x}_3 = (-2, 2, 1, 1)$. Use the Gram-Schmidt process to find an orthonormal basis for W.

4. We proved in lectures that if V is a real inner product space, and $\mathbf{u}, \mathbf{v} \in V$, then the *Cauchy-Schwarz* inequality holds:
   $$|(\mathbf{u}, \mathbf{v})| \leq ||\mathbf{u}|| \cdot ||\mathbf{v}||.$$
   In this question we attempt to establish the identical result when V is a complex inner product space.
   (a) Take the proof in lectures when the field is $\mathbb{R}$, and replace $\mathbb{R}$ by $\mathbb{C}$ throughout. Why is it that the argument is no longer valid?
   (b) Show that the Cauchy-Schwarz inequality holds when $(\mathbf{u}, \mathbf{v}) = 0$, so that we may assume that $(\mathbf{u}, \mathbf{v}) \neq 0$, and so that $\mathbf{u} \neq \mathbf{0}$.
   (c) Suppose that $\lambda \in \mathbb{R}$ and put $\alpha = (\mathbf{u}, \mathbf{v})$. Let $\mathbf{w} = \lambda\mathbf{u} + \alpha\mathbf{v}$. Show that
   $$\lambda^2||\mathbf{u}||^2 + 2\lambda|\alpha|^2 + |\alpha|^2||\mathbf{v}||^2 \geq 0.$$
   (d) Deduce the Cauchy-Schwarz inequality.

Please hand in solutions as directed by your tutor.