Group Theory: Math30038, Sheet 9 Solutions

GCS

1. Suppose that G is a simple group of order 60.

   (a) *Show that G has a subgroup A of order 12.*
   Solution Since G is simple there must be more than one Sylow 2-subgroup of G. Let P and Q be distinct subgroups of G of size 4. If $P \cap Q = T$ has order 2, then T has index 2 in both P and Q and so is normalized by each of them. Let $H = N_G(T)$ which has order more than 4, divisible by 4 and dividing 60. Now $H \neq G$ else $T \trianglelefteq G$. Also $|G : H| \neq 3$ else a Poincaré argument would have 60 dividing $3! = 6$. Thus $|N_G(T)| = 12$. Let $A = N_G(T)$.

   (b) Show that A has exactly 5 different conjugates.
   Solution $|N_G(A)|$ divisible by 12 and divides 60. However, A is not normal in G and so $N_G(A) = A$. Let G act on the conjugates of A via conjugation. The size of the orbit is $|G : N_G(A)| = 5$.

   (c) *Show that there is an injective homomorphism from G to S_5.*
   Solution Let the image of this map be H, a subgroup of index 2 in S_5. Let Ω be the set of conjugates of A, a set of size 5. The conjugation action of G on Ω induces a group homomorphism $\theta : G \to \mathrm{Sym}(\Omega)$. This homomorphism has non-trivial image, and trivial kernel by simplicity of G. This gives a monomorphism from G to $\mathrm{Sym}(\Omega)$. Labelling the elements fo Ω using the first five natural numbers we obtain a monomorphism from G into S_5. The image X is a copy of G and has index 2 in S_5.

   (d) *Show that both A_5 and H both contain every element of S_5 of the form g^2 and therefore every 5-cycle and every 3-cycle.*
   Solution Suppose that Z is any subgroup of S_5 of index 2. Then

Z is normal in S_5 and the quoent group S_5/Z is cyclic of order 2. Thus if $g \in G$, then $(gZ)2 = Z$ so $g^2 \in Z$. Thus all squares are in Z. Now all 3-cycles and 5-cycles are squares because $(a, b, c) = (a, c, b)2$ and $(a, b, c, d, e) = (a, d.b, e, c)^2$.

(e) *Show that $H = A_5$.*
Solution There are 20 three cycles and 24 5-cycles in S_5 so $A_5 \cap H$ is a subgroup of each of A_5 and H and has size at least 44. Therefore $A_5 \cap H$ has order 60 by Lagrange's theorem. Thus $H = A_5$.

(f) *Deduce that any simple group of order 60 must be isomorphic to A_5.*
Solution That is what we have shown.

Incidentally, here is another way to show that G is isomorphic to A_5, and this method cuts out (d), (e) and (f): Now let $Y = X \cap A_5$. Now from an early sheet, $|A_5 : Y| \leq |S_5 : X| = 2$ so either $|A_5 : X|$ is 2 or 1. It cannot eb 2, else then Y would be normal in A_5 which would violate the simplicity of A_5. Therefore the index is 1 so $A_5 \leq Y$. Now each of A_5 and Y has order 60 so $Y = A_5$.

2. Show that the following presentations all describe the trivial group.

   (a) $\langle x \mid x^2, x^3\rangle$,
   Solution Let x denote the image of x in the presented group G. Thus $x = x^3(x^2)^{-1} = 1$. Since G is cyclic and generated by x, then G is trivial.

   (b) $\langle x, y \mid xy = yx, xyx = yxy, x^5, y^6\rangle$
   Solution As before we regard x and y as elements of the presented group G. Now G is abelian since $xy = yx$. Therefore $y = x$ and G is cyclic. Now $1 = x^5 = y^6 = x^6$ so $x = 1 = y$ and G is trivial.

   (c) $\langle x, y \mid x^2, y^4, xyx = y^2, xyxyxy\rangle$.
   Solution Working in the established way in G we discover that $(xyx)^2 = y^4 = 1$. Thus $xyxxyx = 1$ so $xyyx = 1$ so $yy = xx = 1$. Thus $xyx = y^2 = 1$ so $y = xx = 1$. Now $xyxyxy = x^3 = 1 = x^2$ so $x = 1$ in G and G is the trivial group.

3. We discuss the group $G = \langle x, y \mid x^2 = y^2, (xy)^2 = x^2, x^4\rangle$ in stages:

(a) Let $T = \langle x^2 \rangle \leq G$ so $|T| \leq 2$. Show that $T \trianglelefteq G$.
Solution $T = \{1, x^2\} = \{1, y^2\}$. Thus in G the element or elements of T commute with both x and y and therefore with all elements of G. Thus T is central in G and is therefore normal in G.

(b) *Let $H = G/T$. Show that xT and yT commute so that H is abelian.*
Solution Now yxT is the inverse of xyT and so is xyT so $xyT = yxT$. Thus xT and yT commute so G/T is abelian.

(c) *Show that $|H| \leq 4$.*
Solution It is easy to verify that the group generated by xT and yT is $\{T, xT, yT, xyT\}$ and so H has size at most 4.

(d) Deduce that $|G| \leq 8$.
Solution $|G| = |G : T| \cdot |T| \leq 8$.

(e) Find a group of order 8 containing elements a, b such that $a^2 = b2$, $(ab)^2 = a^2$ and $a^4 = 1$. Apply von Dyck's theorem to deduce that G has a homomorphic image of order 8, and by part (d) deduce that this homomorphism is an isomorphism.
Solution The quaterion group will do the trick, in the usual notation i and j satisfy the relations (for x and y respectively). By Von Dyck's theorem there is an epimorphism from G to a group of order 8, so G has size at least 8. Thus G has order 8, and the homomorphism from G onto the quaternion is an isomorphism.

4. *Discuss the (language) group L presented by the small roman alphabet with relations consisting of all homonyms. Thus we have relations such as too = two, practice = practise, sea = see and so on. The group you get in my ideolect is C_2 (South London with a transparent veneer of civilization), whereas in South African English you get the trivial group. On the other hand, I understand that in German the spelling of a word is determined by its pronunciation, so L is a free group. To get you started in L for English, since two = too in L, then $t^{-1}twoo^{-1} = t^{-1}tooo^{-1}$ and thus $w = o$ in L. Also too = to so $o = 1$ and therefore $w = 1$. Here 1 denotes the identity element of L for English.*
Solution Well, this is best done for fun. I reckon South African Englsih can kill off the group with $felt = veldt$.