# October 2004 Exam 1 Outline Solutions

November 13, 2004

1. We are given a family of disks in the plane, with pairwise disjoint interiors. Each disk is tangent to at least six other disks of the family. Show that the family is infinite.

   **Solution 1** Suppose that the family is finite. Let $D(O, r)$ be a smallest disk in the family, and let $D(O_i, r_i)$, $i = 1, \ldots, 6$ be six disks in the family tangent to $D(O, r)$, numbered so that $O_1, \ldots, O_6$ are in that cyclic order around $O$. Since $D(O, r)$ is a smallest disk, $r_i, r_{i+1} \geq r$, for each $i = 1, \ldots, 6$ (taking $O_7 = O_1$, $r_7 = r_1$). So

   $$OO_i = r + r_i, \quad OO_{i+1} = r + r_{i+1}, \quad O_iO_{i+1} \geq r_i + r_{i+1} \geq OO_i, OO_{i+1}.$$

   So in triangle $OO_iO_{i+1}$ no side is longer than $O_iO_{i+1}$. This implies that $\angle O_iOO_{i+1} \geq 60^{\text{o}}$. Equality occurs iff $r = r_i = r_{i+1}$ and $D(O_i, r_i)$ and $D(O_i, r_{i+1})$ are tangent. Since $\angle O_1OO_2 + \angle O_2OO_3 + \cdots + \angle O_6OO_1 = 360^{\text{o}}$, for each $i$ we must have $\angle O_iOO_{i+1} = 60^{\text{o}}$ and $r = r_i$. That is, $D(O, r)$ is surrounded by six disks of radius $r$, tangent in a cycle. Likewise, each of these disks is surrounded by six disks of radius $r$ and so on. So the family contains every disk in an infinite close-packed lattice of disks of radius $r$, a contradiction.

   **Solution 2 (Martin Orr, Konrad Dabrowski)** Euler's Formula states: If a connected planar graph $G$ has $V$ vertices, $E$ edges and $F$ faces (in some planar embedding), then $V - E + F = 2$. As long as $V \geq 3$, each face of (any embedding of) $G$ is bounded by at least 3 edges, and each edge lies in at most two faces, so $F \leq 2E/3$. In Euler's formula this implies that $E \leq 3V - 6$. In particular, the average degree $2E/V$ of any planar graph is less than 6.

In this problem, define a graph $H$ whose vertices are the disks, and in which two vertices form an edge if and only if the corresponding disks are tangent. The graph $H$ is planar – an embedding of it in the plane is obtained by placing a vertex at the centre of each disk, and joining the centres of tangent disks by a straight line segment. So the average degree of $H$ must be less than 6; a contradiction, since each disk is tangent to at least 6 others.

2. Determine all function defined on the positive reals and taking real values which satisfy

$$f(x+y) = f(x^2+y^2) \text{ for all } x, y \in \mathbb{R} \text{ such that } x+y>0.$$

**Solution (William Laffan)** We shall show that the only possibilities are constant functions, $f \equiv c$ for some $c \in \mathbb{R}$. Indeed, suppose $a, b$ are two positive real numbers:

$$\begin{aligned} (x,y) = ((a+b)/2, (a-b)/2) &\implies f(a) = f((a^2+b^2)/2) \\ (x,y) = ((a+b)/2, (b-a)/2) &\implies f(b) = f((a^2+b^2)/2) \end{aligned}$$

So $f(a) = f(b)$, and $f$ is constant. Finally, if $c \in \mathbb{R}$, the function $f(x) = c \ \forall x$ does satisfy the conditions of the problem.

3. Let $a$ and $b$ be integers. Is it possible to find integers $p$ and $q$ such that $p + na$ and $q + nb$ are coprime for all integers $n$?

**Solution** It is always possible to find such integers $p, q$. If $a = b = 0$ we may take $p = q = 1$. Suppose then that $(a, b) \neq (0, 0)$. Let $(a, b) = h$, and set $a = a'h$, $b = b'h$. The integers $a'$ and $b'$ are coprime, so there are integers $p$ and $q$ such that $b'p - a'q = 1$. And now for these $p, q$ and any $n \in \mathbb{Z}$ we have

$$b'(p+na) - a'(q+nb) = (b'p - a'q) + (b'na'h - a'nb'h) = 1.$$

So $p + na$ and $q + nb$ are coprime.