October 2004 Exam 2 Outline Solutions

November 13, 2004

1. Find the smallest positive integer n such that

   (a) n has exactly 144 distinct positive divisors and

   (b) there are 10 consecutive positive integers among the divisors of n.

   Solution Among any 10 consecutive positive integers, at least one is divisible by 8, at least one by 9, at least one by 5, and at least one by 7. So if there are 10 consecutive positive integers among the divisors of n, then n is divisible by $k = 2^3 \cdot 3^2 \cdot 5 \cdot 7$. Conversely, any positive integer divisible by k has 10 consecutive integer divisors - namely $1, 2, \ldots, 10$. Now if the prime factorization of n is $n = 2^{\alpha_1} 3^{\alpha_2} \cdots p_k^{\alpha_k}$ the number of divisors of n (usually denoted $d(n)$ in the literature) is given by $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$. In our case, if n had more than 5 distinct prime factors, we'd have $d(n) \geq 4 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 192$. So n has at most 5 distinct prime factors; and since we seek the smallest n satisfying the conditions in the question, if n has a fifth prime factor (after $2, 3, 5, 7$), then it is 11. There are now a few case to check. We know that $n = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} 7^{\alpha_4} 11^{\alpha_5}$, where $\alpha_1 \geq 3$, $\alpha_2 \geq 2$, $\alpha_3 \geq 1$ and $\alpha_4 \geq 1$, $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4 \geq \alpha_5$ and $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)(\alpha_4 + 1)(\alpha_5 + 1) = 144$. These imply that $\alpha_1 \leq 11$, leaving the following possibilities (in a complete solution you should use the conditions above to show carefully that these are the only possibilities):

$$
\begin{aligned}
(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (11, 2, 1, 1, 0) &\quad\rightarrow\quad n = 256k \\
(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (8, 3, 1, 1, 0) &\quad\rightarrow\quad n = 96k \\
(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (7, 2, 2, 1, 0) &\quad\rightarrow\quad n = 80k \\
(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (5, 2, 1, 1, 1) &\quad\rightarrow\quad n = 44k \\
(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (5, 3, 2, 1, 0) &\quad\rightarrow\quad n = 60k \\
(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (5, 5, 1, 1, 0) &\quad\rightarrow\quad n = 108k \\
(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (3, 2, 2, 1, 1) &\quad\rightarrow\quad n = 55k \\
(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (3, 3, 2, 2, 0) &\quad\rightarrow\quad n = 105k
\end{aligned}
$$

So the smallest is $n = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 = 110880$.

2. We are given six straight lines in space. Among any collection of three of those lines, at least one pair is perpendicular. Show that the given lines can be labelled $l_1, l_2, \dots, l_6$ in such a way that l_1, l_2, l_3 are pairwise perpendicular, and l_4, l_5, l_6 are pairwise perpendicular.

   Solution (Adam Bull) You may need some coloured pens to follow this solution properly. Define a graph G with vertices $v_1, v_2, \dots, v_6$. Vertices v_i and v_j are joined by a blue edge if l_i and l_j are perpendicular, and by a red edge if l_i and l_j are not perpendicular. Since the Ramsey number $R(3)$ is 6, the graph G contains a monochromatic triangle. Because some two of any three lines are perpendicular, there is no red triangle, so there must be a blue triangle. Wlog wma v_1, v_2, v_3 are the vertices of a blue triangle. If $v_4v_5v_6$ is also a blue triangle, we are done. Suppose then that this is not the case. Wlog wma v_4v_5 is red. Now each of the triangles $v_1v_4v_5$, $v_2v_4v_5$, $v_3v_4v_5$ contains at least one blue edge. So at least one of the vertices v_4 and v_5 is joined by blue edges to at least two of v_1, v_2, v_3. Wlog wma v_4 is joined by blued edges to v_1 and v_2. The lines l_1 and l_2 are perpendicular. So the lines l_3 and l_4 (each of which is perpendicular to both l_1 and l_2) are parallel. So the edge v_3v_4 is red. Also, the fact that v_4v_5 is red and that l_4 is parallel to l_3 imply that v_3v_5 is red. So now $v_3v_4v_5$ is a red triangle. This is a contradiction.

3. Find the greatest possible value of the expression

   $$(a+b)^4 + (a+c)^4 + (a+d)^4 + (b+c)^4 + (b+d)^4 + (c+d)^4$$

   given that the real numbers a, b, c and d satisfy

   $$a^2 + b^2 + c^2 + d^2 \leq 1.$$

   Solution By multiplying expressions out, it is easy to check that

   $$(a+b)^4 + (a+c)^4 + (a+d)^4 + (b+c)^4 + (b+d)^4 + (c+d)^4$$

   is equal to

   $$6(a^2+b^2+c^2+d^2)^2 - \{(a-b)^4+(a-c)^4+(a-d)^4+(b-c)^4+(b-d)^4+(c-d)^4\}.$$

   Therefore

   $$(a+b)^4 + (a+c)^4 + (a+d)^4 + (b+c)^4 + (b+d)^4 + (c+d)^4 \leq 6$$

   with equality if and only if $a = b = c = d = 1/2$.