UK IMO Squad, December 2004

Exam 1

4 hours and 30 minutes

Please bring your script with you to Heathrow Airport if you are travelling to Budapest. If not, please post your solution to `Dr G C Smith, UK IMO Squad December Exam, Department of Mathematics, University of Bath, Claverton Down, Bath BA2 7AY` *by the end of 2004.*

1. The numbers $1, 2, \ldots, 100$, each written 100 times, are arranged in a 100×100 table. Prove that there is a row or a column containing at least 10 different numbers.

2. Let a, b and c be positive real numbers such that $abc \geq 1$. Prove that

$$a^3 + b^3 + c^3 \geq ab + bc + ca.$$

3. A convex polygon has 2004 vertices, no four of which are concyclic. A triangle with vertices drawn from these points is called *thick* if all other 2001 vertices are in the interior of its circumcircle, or *thin* if all other 2001 vertices are outside its circumcircle. Prove that the number of thick triangles is equal to the number of thin triangles.