UK IMO Squad, December 2004

Exam 2

4 hours and 30 minutes

Please bring your script with you to Heathrow Airport if you are travelling to Budapest. If not, please post your solution to `Dr G C Smith, UK IMO Squad December Exam, Department of Mathematics, University of Bath, Claverton Down, Bath BA2 7AY` *by the end of 2004.*

1. An "infinite chessboard" has finitely many of its 1×1 squares coloured black. The rest are white. Prove that it is possible to select a finite number of squares (with edges coincident with sides of the given 1×1 squares) such that each of the following conditions is satisfied.

   (a) The interiors of different squares are disjoint.

   (b) Every black square is contained in a selected square.

   (c) The number of black 1×1 squares in each selected square is at least $\frac{1}{5}$, and at most $\frac{4}{5}$ of the total number of 1×1 squares in the square.

2. Prove that there is no integer $n > 1$ such that n divides $3^n - 2^n$.

3. The edge SA of the tetrahedron $SABC$ is perpendicular to the plane of $\triangle ABC$. Two different spheres σ_1 and σ_2 each pass through A, B and C and are internally tangent to a sphere σ with centre S. The radii of σ_1 and σ_2 are r_1 and r_2, respectively. Determine the radius of σ.