November Exam 1

1. Let $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ be positive integers such that the sums $x_1 + x_2 + \cdots + x_m$ and $y_1 + y_2 + \cdots + y_n$ are equal and less than mn. Prove that in the equality

$$x_1 + x_2 + \cdots + x_m = y_1 + y_2 + \cdots + y_n$$

one can cancel some terms and obtain another equality.

('cancel some terms' should be understood not to include the possibility of cancelling *no* terms or *all* terms in the equality.)

2. Let ABC be a triangle and D, E, F the respective midpoints of the sides BC, CA, AB. Congruent circles S_A, S_B and S_C are centred at A, B and C respectively. The circle S_A cuts the line EF at X_1 and X_2; the circle S_B cuts the line FD at Y_1 and Y_2; and the circle S_C cuts the line DE at Z_1 and Z_2. Show that the six points $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ are concyclic.

3. I have a machine which shuffles a deck of 53 cards. My machine does not shuffle at random; in fact it came with a guarantee that it would permute the 53 cards in exactly the same way every time. If I use the machine to shuffle 53 cards over and over again, (and I am very patient), I will eventually get back to the original order. What is the largest possible number of shuffles that it could take before the deck first returns to its original order?

Solutions should be sent to Adrian Sanders, Trinity College, Cambridge CB2 1TQ for receipt on or before Wednesday 24 November.