NOVEMBER 04 **TBL**

UKMT ADVANCED MENTORING SCHEME — QUESTIONS — SHEET 1

1. For which integers n does there exist a polynomial with integer coefficients with $p(1) = 7$ and $p(8) = n$?

2. Suppose $P(x) = ax^2 + bx + c$ is a quadratic with nonnegative real coefficients. Show that, for any positive real x,

$$P(x)P(x^{-1}) \geq (P(1))^2$$

3. Determine all pairs (a, b) of integers such that

$$(a^3 + b)(a + b^3) = (a + b)^4$$

4. Let $\mathbb{R}^+$ denote the set of all strictly positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all pairs (x, y) of positive reals, we have that

$$x^2(f(x) + f(y)) = (x + y)f(f(x)y)$$

5. Prove that the equation

$$x^2 + y^2 + z^2 + w^2 = 2^{2004}$$

has exactly two solutions in the set of integers.

6. Let n be a positive integer, and let $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ be complex numbers such that for every choice of $\epsilon_1, \ldots, \epsilon_n$ from the set $-1, 1$ we have that

$$|\epsilon_1 z_1 + \cdots + \epsilon_n z_n| \leq |\epsilon_1 w_1 + \cdots + \epsilon_n w_n|$$

Prove that

$$|z_1|^2 + \cdots + |z_n|^2 \leq |w_1|^2 + \cdots + |w_n|^2$$

7. Find all non-negative integers k such that we can find non-negative integers a and b with $ab \neq 1$ such that

$$\frac{a^2 + ab + b^2}{ab - 1} = k$$

8. We have a square $ABCD$ lying inside a circle γ. We construct a circle γ_A as follows: γ_A lies inside the angle opposite to $\angle BAD$, and is tangent to γ, AD produced, and AB produced. (So γ_A lies entirely *outside* the square $ABCD$.) Let A_1 be the point of tangency of γ_A and γ. Define B_1, C_1, D_1 similarly. Show that AA_1, BB_1, CC_1 and DD_1 are concurrent.

9. Let a, b, c be integers with b odd. Consider the sequence (x_n) satisfying $x_0 = 4, x_1 = 0, x_2 = 2c, x_3 = 3b$ and

$$x_n = ax_{n-4} + bx_{n-3} + cx_{n-2} \qquad \text{for } n \geq 4$$

Show that for p prime and m an integer, we have x_{p^m} is divisible by p.