# Advanced Mentoring Scheme

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1. A natural number is said to be 3-partite if its divisors can be partitioned into 3 subsets, each of which has the same sum.
   (a) Find a 3-partite number.
   (b) Show that there infinitely many 3-partite numbers.

2. In the usual notation for triangle $ABC$, show that
$$\frac{\cos A}{a^3} + \frac{\cos B}{b^3} + \frac{\cos C}{c^3} \geq \frac{3}{2abc}.$$

3. Prove that the medians from vertices $A$ and $B$ of triangle $ABC$ are mutually perpendicular if and only if
$$|BC|^2 + |AC|^2 = 5|AB|^2.$$

4. Prove the inequality
$$\frac{a^2}{(a+b)(a+c)} + \frac{b^2}{(b+a)(b+c)} + \frac{c^2}{(c+a)(c+b)} \geq \frac{3}{4}$$
holds for all positive real numbers $a$, $b$ and $c$.

5. Suppose that $x, y$ and $z$ are real numbers such that $xyz = -1$. Show that
$$x^4 + y^4 + z^4 + 3(x+y+z) \geq \frac{x^2}{y} + \frac{x^2}{z} + \frac{y^2}{x} + \frac{y^2}{z} + \frac{z^2}{x} + \frac{z^2}{y}.$$

6. Suppose that $\angle A$ is the smallest of the three angles of triangle $ABC$. Let $D$ be a point on the arc $BC$ of the circumcircle of $ABC$ which does not contain $A$. Let the perpendicular bisectors of $AB$, $AC$ intersect $AD$ at $M$ and $N$ respectively. Let $BM$ and $CN$ meet at $T$. Prove that $BT + CT \leq 2R$ where $R$ is the circumradius of triangle $ABC$.

7. The sequences of real numbers $(x_n)$, $(y_n)$ and $(z_n)$ are defined by initial data $x_1 = 2$, $y_1 = 4$ and $z_1$ such that $x_1 y_1 z_1 = x_1 + y_1 + z_1$, and recurrences

$$x_{n+1} = \frac{2x_n}{x_n^2 - 1}, \quad y_{n+1} = \frac{2y_n}{y_n^2 - 1}, \quad z_{n+1} = \frac{2z_n}{z_n^2 - 1}.$$

   (a) Prove that the recurrences are legitimate, in that $x_n^2, y_n^2$ and $z_n^2$ are never 1.

   (b) Is there a positive integer $k$ such that $x_k + y_k + z_k = 0$?

8. Find all real numbers $\alpha$ with the property that every member of the sequence

$$\cos\alpha,\ \cos 2\alpha,\ \cos 2^2\alpha, \ldots,\ \cos 2^n\alpha,\ \ldots$$

is negative. *Note that it is implicit that we are using radians.*

9. Let triangle $ABC$ have side lengths $a, b$ and $c$ as usual. Points $P$ and $Q$ lie inside this triangle and have the properties that

$$\angle BPC = \angle CPA = \angle APB = 120^o$$

and

$$\angle BQC = 60^0 + \angle A, \quad \angle CQA = 60^o + \angle B, \quad \angle AQB = 60^0 + \angle C.$$

Prove that

$$(|AP| + |BP| + |CP|)^3 \cdot |AQ| \cdot |BQ| \cdot |CQ| = (abc)^2.$$

10. There are 2000 white balls in a box, and an ample supply of green, red and more white balls is available. The following operations are allowed on the contents of the box:

   (a) Replace two white balls by a green ball.

(b) Replace two red balls by a green ball.

(c) Replace two green balls by a white ball and a red ball.

(d) Replace a white ball and a green ball by a red ball.

(e) Replace a green ball and a red ball by a white ball.

After finitely many operations there are three balls in the box. Prove that at least one of them must be green. Is it possible to have only one ball left in the box after finitely many operations?

11. Let $S$ be a non-empty finite set (an *alphabet*). A *word* is a finite sequence of elements of $S$. We are given a finite collection of $m$ forbidden words, and an infinite sequence $a_1, a_2, a_3, \ldots$ of elements of the alphabet so that no forbidden word appears as a block (i.e. a finite consecutive sequence of letters). Show that there exists a two sided infinite sequence of elements of the alphabet

$$\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots$$

such that none of the forbidden words appears as a block.

12. Find all positive real numbers $x$ such that

$$\sqrt{x} + \sqrt[3]{x+7} = \sqrt[4]{x+80}.$$

13. The points $M$ and $N$ are the points of tangency of the incircle of the isosceles triangle $ABC$ which are on the sides $AC$ and $BC$. The sides of equal length are $AC$ and $BC$. A tangent line $t$ is drawn to the minor arc $MN$. Suppose that $t$ intersects $AC$ and $BC$ at $Q$ and $P$ respectively. Suppose that the lines $AP$ and $BQ$ meet at $T$.

(a) Prove that $T$ lies on the line segment $MN$.

(b) Prove that the sum of the areas of triangles $ATQ$ and $BTP$ is minimized when $t$ is parallel to $AB$.

14. There are $n \geq 4$ distinct points in the plane such that the distance between each pair of them is an integer. Prove that at least $1/6$ of these distances are divisible by 3.

15. Let $A$ be a finite subset of the prime numbers, and $a$ be a positive integer. Show that only finitely many positive integers $m$ are such that all prime divisors of $a^m - 1$ are in $A$.