October 2005 UK IMO Exam 1

4 hours 30 minutes

1. Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that

   (a) $f(0) = 0, \ f(1) = 1$;

   (b) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a, b such that $a < 1 < b$.

2. Let Γ_1, Γ_2, Γ_3 and Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C and D, and that these points are different from P.

   Prove that
   $$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

3. Each positive integer a (written in base 10 notation) undergoes the following procedure in order to obtain the number $d = d(a)$:

   (a) move the last digit of a to the first position to obtain the number b;

   (b) square b to obtain the number c;

   (c) move the first digit of c to the end to obtain the number d.

   (For example, for $a = 2003$, we get $b = 3200$, $c = 10240000$, and $d = 02400001 = 2400001 = d(2003)$.)

   Find all numbers a for which $d(a) = a^2$.