October 2005 UK IMO Exam 2

4 hours 30 minutes

1. Let m be a fixed integer greater than 1. The sequence $x_0, x_1, x_2, \ldots$ is defined as follows:

$$x_i = \begin{cases} 2^i & \text{if } 0 \leq i \leq m-1; \\ \sum_{j=1}^{m} x_{i-j} & \text{if } i \geq m. \end{cases}$$

Find the greatest k for which the sequence contains k consecutive terms divisible by m.

2. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be real numbers. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be the matrix with entries

$$a_{ij} = \begin{cases} 1, & \text{if } x_i + y_j \geq 0; \\ 0, & \text{if } x_i + y_j < 0. \end{cases}$$

Suppose that B is an $n \times n$ matrix with entries $0, 1$ such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A = B$.

3. Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E respectively. The line through P parallel to AB meets CA and CB at F and G respectively. Prove that the lines DF and EG intersect on the circumcircle of triangle ABC.