OCTOBER 05 **TBL**

UKMT ADVANCED MENTORING SCHEME — QUESTIONS — SHEET 1

1. Let n be a fixed positive integer. We are given a 2^n by 2^n chessboard, one of whose squares has been marked, and an infinite supply of L-shaped pieces, each of which is just the right size to cover three adjacent squares. Show that you can always tile the unmarked squares of the chessboard with the pieces.

2. There is a circular road in the desert, N miles long, with n bases placed (possibly with uneven spacing) around the road. Each base has a certain quantity of fuel in its stores: there is a total of N gallons of fuel. You have a jeep, which does precisely one mile to the gallon. You start at one of the bases b, putting all the fuel from that base in your jeep, and drive clockwise around the road. Whenever you get to a base, you use the fuel there to refuel the jeep. Show that, for some choice of b, you will not run out of fuel until you return to b having completed a circuit.

3. A large cube consisting of n^3 small cubes sits on the table. We wish to remove small cubes one by one until no small cubes are left, but we can only remove cubes whose upper face is visible. What is the smallest n for which we can do this in more than 10^{18} ways?

4. Two squares are erected outwardly on the sides BC and CA of a triangle ABC. Let P and Q be the midpoints of these squares, and let M be the midpoint of AB. Prove that MPQ is an isosceles right angled triangle.

5. Find all triples (x, y, z) of integers satisfying

$$(x + y + z)^3 = 4(x^3 + y^3 + z^3) + 12xyz + 9$$

6. Let a, b, c be positive reals with $abc = 1$. Prove that

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \leq 1$$

7. Consider the set

$$S = \{(x, y) : x, y \in \mathbb{Z}, 1 \leq x \leq 4, 1 \leq y \leq 4\}$$

of 16 points in the plane. Show that one can choose 6 of these points so that it is not possible to draw an isosceles triangle all of whose vertices lie amongst the points chosen. Show also that given any seven of the points of S, it *is* possible to draw an isosceles triangle all of whose vertices lie amongst the points given.

8. There are n knights at Camelot; given any pair of knights, they are either *friends* or *enemies*. Each knight has more friends than enemies; show that the knights may be arranged around a circular table such that no two enemies are adjacent.