Erratum: Abelian surfaces in toric 4-folds

T. Kajiwara has pointed out an error of calculation on page 414 of [1], which seriously affects the results of the paper. In the calculations of the intersection numbers on $X = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(\kappa_1) \oplus \mathcal{O}_{\mathbb{P}^2}(\kappa_2)\big)$, the formula $\mathbf{b}^4 = \kappa^2$ should read $\mathbf{b}^4 = \kappa^2 - \kappa_1\kappa_2$. Consequently, the version of the self-intersection formula given on the same page as equation (2) is also incorrect. The correct formula is

$$(3 - 3\kappa + \kappa_1\kappa_2)\nu + (9 - 2\kappa)\mu + 3\lambda = 2\lambda\nu - 2\kappa\mu\nu + \mu^2 + \kappa_1\kappa_2\nu^2. \qquad (2')$$

The proof of Theorem 1.4 given in [1] has to be modified but the statement remains true: indeed, much more is true. Section 2 is entirely independent and is not affected by the error. However, Proposition 3.8 is wrong and as a result Theorem 3.1 is also wrong. The example given in Section 3 was supposedly chosen so as to arrange for the double point scheme to have length zero; but this was incorrectly done and therefore the morphism ϕ constructed in Section 3 is not an embedding. The rest of the paper is unaffected: in particular ϕ is birational onto its image (Corollary 3.5) and the image has only isolated singularities (Corollary 3.7)

Kajiwara observes that by using the methods of Section 1 of [1] and results of Batyrev, it can readily be shown that there is no nondegenerate embedding from an abelian surface to a pseudo-Del Pezzo or Del Pezzo 4-fold, apart from the previous known cases of $\mathbb{P}^4$ and $\mathbb{P}^3 \times \mathbb{P}^1$.

In fact, although the equation (2') does have integer solutions, there is no way to construct an example of the type claimed in the paper. This is because the inequality (4), which says that $E.D_1 \geq 0$, is weaker than what we really need. The inequalities (1) and (3) can only be satisfied with equality in (3), meaning that D_1 is a union of disjoint elliptic curves. Moreover D_1 moves, so it cannot be a single elliptic curve. But then the polarisation E, which is not a product polarisation, must have degree at least 2 on each component of D_1, and degree 3 at least if it is very ample. However, it turns out that (1), (2') and (3) imply that $\mu - \kappa_1\nu \leq 4$ with equality only in one case ($\kappa_1 = \kappa_2 = 1$, $\lambda = 22$, $\mu = 18$, $\nu = 14$) in which E would have to be very ample anyway.

Reference

[1] G.K. Sankaran, *Abelian surfaces in toric 4-folds*, Math. Ann **313** (1999), 409–427.