ANISOTROPIC MESH REFINEMENT, THE CONDITIONING OF GALERKIN BOUNDARY ELEMENT MATRICES AND SIMPLE PRECONDITIONERS*

IVAN G. GRAHAM† AND WILLIAM MCLEAN‡

Abstract. In this paper we obtain upper and lower bounds on the spectrum of the stiffness matrix arising from a finite element Galerkin approximation (using nodal basis functions) of a bounded, symmetric bilinear form which is elliptic on a Sobolev space of real index $m \in [-1, 1]$. The key point is that the finite element mesh is required to be neither quasiuniform nor shape regular, so that our theory allows anisotropic meshes often used in practice. (However, we assume that the polynomial degree of the elements is fixed.) Our bounds indicate the ill-conditioning which can arise from anisotropic mesh refinement. In addition we obtain spectral bounds for the diagonally scaled stiffness matrix, which indicate the improvement provided by this simple preconditioning. For the special case of boundary integral operators on a $2D$ screen in $\mathbb{R}^3$, numerical experiments show that our bounds are sharp. We find that diagonal scaling essentially removes the ill-conditioning due to mesh degeneracy, leading to the same asymptotic growth in the condition number as arises for a quasi-uniform mesh refinement. Our results thus generalise earlier work by Bank and Scott (1989) and Ainsworth, McLean and Tran (1999) for the shape-regular case.

Keywords: Symmetric elliptic problems, boundary integral equations, Galerkin approximation, anisotropic refinement, spectral bounds, diagonal scaling, condition number estimates.
AMS Subject classifications: 65N38, 65N30, 65N22, 65F10, 65F35

1. Introduction. Edge and corner singularities are characteristic features of solutions to 3D elliptic boundary value problems and, in both finite element and boundary element methods, are commonly dealt with by some kind of local mesh refinement. Typically, an edge singularity is strongly *anisotropic*: the lack of smoothness occurs only in directions normal to the edge. For this reason, the local mesh refinement should also be anisotropic, if we are to minimise the number of degrees of freedom used to achieve a sufficiently small error in, say, the energy norm. Extra refinement is not needed parallel to an edge, except maybe in the vicinity of a corner. The meshes that result from such local refinement are certainly not quasi-uniform, and usually even fail to be *shape-regular* because elements near edges but away from any corner may have a very large aspect ratio.

In this paper we investigate the influence of such meshes on the condition number of the stiffness matrix arising in the Galerkin approximation of a class of symmetric elliptic problems. Our general framework includes as special cases the single layer and the hypersingular boundary integral equations for the Laplacian on the surface of a 3D Lipschitz domain or on a Lipschitz screen as well as the Dirichlet problem for second-order symmetric elliptic PDEs. We obtain general bounds for the spectrum of the resulting Galerkin matrix in terms of quantities which depend on the geometry of the elements and the particular basis functions utilised.

In addition we study in detail two model problems – the weakly singular and hypersingular equations for the Laplacian on a rectangular screen, discretised using classical tensor-product power-graded meshes. For these model problems our general estimates yield explicit bounds in terms of the number of degrees of freedom and the strength of the power grading. We show by numerical experiments that our estimates are sharp and, moreover, exhibit a strong increase in the condition number as the maximum aspect ratio of the elements increases.

These results have practical implications for the performance of iterative techniques such as conjugate gradients, which are often used as solvers for the dense linear systems which arise in these methods (usually combined with a fast matrix-vector multiplication such as Fast Multipole [12] or Panel-Clustering [8]). Efficient solvers require effective preconditioners and as a first step in this direction we also analyse in detail the use of diagonal scaling. We obtain general estimates for the spectrum of the diagonally scaled matrix and again investigate this in fine detail for the special cases of the model problems mentioned above.

Throughout the paper Γ will denote either a bounded, d-dimensional Lipschitz surface in $\mathbb{R}^{d+1}$, for $d = 2$, or else a bounded Lipschitz domain in $\mathbb{R}^d$, for $d = 2$ or 3. In the former case, the surface Γ may be open or closed. B will denote a bounded and *symmetric* bilinear form such that, for some Sobolev index m satisfying $|m| \le 1$, $B : \widetilde{H}^m(\Gamma) \times \widetilde{H}^m(\Gamma) \to \mathbb{R}$, and

$$c\|v\|^2_{\widetilde{H}^m(\Gamma)} \le B(v, v) \le C\|v\|^2_{\widetilde{H}^m(\Gamma)} \quad \text{for all } v \in \widetilde{H}^m(\Gamma), \tag{1.1}$$

*Supported by the UK Engineering and Physical Sciences Research Council Grant GR/S43399/01. We thank Dr E. Georgoulis for very helpful comments.

†Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, U.K.
‡School of Mathematics, University of New South Wales, Sydney, NSW 2052, Australia.

where c and C are positive constants. Thus the energy space for B is equivalent to the Sobolev space $\widetilde{H}^m(\Gamma)$. (Here we are working with standard Sobolev spaces on Γ, see §3 for more detail.)

We shall consider approximations of the following variational problem: find $u \in \widetilde{H}^m(\Gamma)$ such that

$$B(u, v) = \langle f, v\rangle_\Gamma \quad \text{for all } v \in \widetilde{H}^m(\Gamma), \tag{1.2}$$

where, with $d\sigma$ denoting the usual surface element on Γ,

$$\langle f, v\rangle_\Gamma = \int_\Gamma f v \, d\sigma.$$

By the Lax–Milgram Lemma, (1.2) has a unique solution $u \in \widetilde{H}^m(\Gamma)$ for each $f \in H^{-m}(\Gamma)$.

Within this abstract framework we can treat not only some boundary element methods, in particular with $m = \pm 1/2$, but also finite element methods for symmetric H^1 elliptic PDEs with homogeneous Dirichlet boundary conditions.

To approximate the solution u of (1.2), we introduce a finite dimensional subspace $X \subseteq \widetilde{H}^m(\Gamma)$ and then apply Galerkin's method, seeking $u_X \in X$ such that

$$B(u_X, v) = \langle f, v\rangle_\Gamma \quad \text{for all } v \in X. \tag{1.3}$$

In this paper we are concerned only with the h-version of the finite element method in which X is a space of piecewise polynomials of fixed degree with respect to a family of increasingly refined partitions (or meshes) $\{\mathcal{P}\}$ on Γ. The partition $\mathcal{P}$ contains elements $K \in \mathcal{P}$ which have diameter h_K and diameter of largest inscribed ball ρ_K. We will introduce a basis for X consisting of nodal basis functions $\{\phi_j : j \in \mathcal{N}\}$ where $\mathcal{N}$ is a suitable index set with cardinality N. We will define the allowable partitions, elements and basis functions more precisely in §3. Writing $u_X = \sum_{k\in\mathcal{N}} \alpha_k \phi_k$, inserting into (1.3) and choosing $v = \phi_j$ for each $j \in \mathcal{N}$ leads to the $N \times N$ linear system

$$\boldsymbol{B}\boldsymbol{\alpha} = \boldsymbol{f}, \tag{1.4}$$

with a symmetric positive definite matrix $\boldsymbol{B} = [B(\phi_k, \phi_j)]$, a solution vector $\boldsymbol{\alpha} = [\alpha_k]$ and a right-hand side vector $\boldsymbol{f} = [\langle f, \phi_j\rangle]$.

The conditioning of $\boldsymbol{B}$ in the case of shape-regular mesh refinement (i.e., $h_K \lesssim \rho_K$ for all $K \in \mathcal{P}$) was investigated by Ainsworth, McLean and Tran in [1, 2], where the condition number estimate

$$\operatorname{cond}(\boldsymbol{B}) \lesssim \left(\frac{h_{\max}}{h_{\min}}\right)^{d-2m} N^{2|m|/d} \quad \text{for } 2|m| < d, \tag{1.5}$$

was proved. Here, $h_{\max} = \max_{K\in\mathcal{P}} h_K$ and $h_{\min} = \min_{K\in\mathcal{P}} h_K$. (For matrices $\boldsymbol{B}$ with real spectrum, $\operatorname{cond}(\boldsymbol{B}) := \lambda_{\max}(\boldsymbol{B})/\lambda_{\min}(\boldsymbol{B})$, where $\lambda_{\max}(\boldsymbol{B})$ and $\lambda_{\min}(\boldsymbol{B})$ denote the largest and smallest eigenvalues of $\boldsymbol{B}$, respectively. The symbols $\lesssim$ and $\simeq$ indicate (in)equality up to a hidden constant, independent of the mesh — see §3.) In the limiting cases $2m = -d$ and $2m = d$ an additional logarithmic factor occurs in the bound (1.5).

For quasi-uniform meshes, $h_{\max} \simeq h_{\min} \simeq h$ so the bound (1.5) gives the well-known result that $\operatorname{cond}(\boldsymbol{B}) = O(N^{2|m|/d}) = O(h^{-2|m|})$. However, this bound deteriorates if the global mesh ratio $h_{\max}/h_{\min}$ becomes large, and the deterioration is stronger the more negative the Sobolev index m. Fortunately, such additional growth in the condition number is easily eliminated by *diagonal scaling*. In fact, let $\boldsymbol{D}$ denote the diagonal matrix formed from $\boldsymbol{B}$ by setting all the off-diagonal entries to zero, and put

$$\boldsymbol{B}' = \boldsymbol{D}^{-1/2}\boldsymbol{B}\boldsymbol{D}^{-1/2}. \tag{1.6}$$

Then it is shown in [1] that in the shape-regular case we have

$$\operatorname{cond}(\boldsymbol{D}^{-1}\boldsymbol{B}) = \operatorname{cond}(\boldsymbol{B}') \lesssim N^{2|m|/d} \quad \text{for } 2|m| < d. \tag{1.7}$$

We remark that $\boldsymbol{B}' = [B(\phi'_j, \phi'_k)]$ is just the Galerkin matrix that arises if we scale the nodal basis so that $\phi'_j = \phi_j/\sqrt{B(\phi_j, \phi_j)}$ has unit energy: $B(\phi'_j, \phi'_j) = 1$.

This paper obtains bounds analogous to (1.5) and (1.7) for the case when the $\{\mathcal{P}\}$ is no longer required to be shape-regular, and each partition $\mathcal{P}$ may contain elements K for which the aspect ratio h_K/ρ_K approaches infinity as the mesh is refined.

In particular we show that in the case of the weakly singular and hypersingular boundary integral operators (and except possibly for some logarithmic factors), diagonal scaling removes the ill-conditioning produced by the high aspect ratios, restoring the rate of growth of the condition number (in terms of the number of degrees of freedom) to essentially what it would be for a quasi-uniform mesh with the same number of degrees of freedom.

We remark that our results not only generalise the results [1, 2] to more general meshes, but they also generalise some of the earlier results of Bank and Scott [3], who obtained the analogous result for H^1 finite elements and shape-regular mesh sequences.

The layout of this paper is as follows. In §2 we motivate the theory by describing the results for the weakly singular and hypersingular operators in detail, without proofs. In §3 we set the theoretical scene by describing the class of finite (boundary) elements which we shall consider (which allow degenerate meshes), and we introduce the corresponding nodal bases. A key step in the theory in [1, 2] is the proving of estimates for Sobolev norms of nodal basis functions. In §4 we extend these estimates to the case of non-shape regular meshes. Here we make essential use of recently derived inverse estimates for finite element functions on anisotropic meshes [7]. In §5 we obtain general bounds on the spectra of $\boldsymbol{B}$ and $\boldsymbol{B}'$ in terms of the geometry of the elements and the Sobolev norms of the nodal basis functions. For the case of power graded meshes and the weakly singular and hypersingular operators these lead to quantitative spectral estimates which are tested in the numerical experiments in §6. These eperiments show that the results for $\boldsymbol{B}'$ are not completely sharp. Sharper results for special cases, which explain the numerical results, are proved in §7. Finally, §8 presents some additional numerical results using a more complicated family of meshes.

2. Examples.

2.1. Two integral equations. The *weakly-singular* (or *single-layer*) boundary integral equation:

$$\frac{1}{4\pi}\int_\Gamma \frac{u(y)}{|x-y|}\,d\sigma_y = f(x) \quad \text{for } x\in\Gamma, \tag{2.1}$$

arises, for example, in the solution of the Dirichlet problem for Laplace's equation in the region exterior to Γ. This equation (2.1) may be written in the form (1.2), with

$$B(u,v) = \frac{1}{4\pi}\iint_{\Gamma\times\Gamma} \frac{u(y)v(x)}{|x-y|}\,d\sigma_x\,d\sigma_y\ . \tag{2.2}$$

Then B satisfies the norm equivalence (1.1) for $m=-1/2$.

The *hypersingular* integral equation,

$$-\frac{1}{4\pi}\int_\Gamma\left(\frac{\partial}{\partial\nu_x}\frac{\partial}{\partial\nu_y}\frac{1}{|x-y|}\right)u(y)\,d\sigma_y = f(x) \quad \text{for } x\in\Gamma, \tag{2.3}$$

arises, for example, in the solution of the Neumann problem for Laplace's equation. (Here $\partial/\partial\nu_x$ denotes the normal derivative at $x\in\Gamma$ and the integral is defined as the finite part integral in the sense of Hadamard.) The integration by parts procedure of Nedelec [10], [9, Theorem 9.15] allows us to write the associated bilinear form as:

$$B(u,v) = \frac{1}{4\pi}\iint_{\Gamma\times\Gamma}\frac{\mathbf{curl}_\Gamma u(x)\cdot\mathbf{curl}_\Gamma v(y)}{|x-y|}\,d\sigma_x\,d\sigma_y\ , \tag{2.4}$$

where $\mathbf{curl}_\Gamma$ denotes the surface curl operator. The norm equivalence (1.1) holds for $m=+1/2$. If the surface Γ is flat then $\mathbf{curl}_\Gamma$ can be replaced by the 2D gradient operator ∇.

It is well-known that the solutions of equations (2.1) and (2.3) in general exhibit singular behaviour near the edges and corners of Γ. In particular near the interior of an edge, the solution u of (2.1) typically has a singularity of order $O(\rho^{\alpha-1})$ as $\rho\to 0$, where ρ is the distance of a point from the edge and $\alpha>1/2$ depends on the angle subtended by the boundary Γ near the edge. The solution of equation (2.3) typically has a singularity of order $O(\rho^\alpha)$. More complicated behaviour appears near corners. The full detail is well-known, see, e.g. [4, 5, 11].

2.2. Power-graded meshes. For the h-version of the boundary integral method, it is common to approximate (2.1) and (2.3) using power graded meshes. To describe these, first consider the special case of a flat, square screen

$$\Gamma = \{\, x\in\mathbb{R}^3 : 0<x_1<1, 0<x_2<1, x_3=0\,\}, \tag{2.5}$$

and think of Γ as a subset of $\mathbb{R}^2$ by writing $x = (x_1, x_2) = (x_1, x_2, 0)$ for $x \in \Gamma$. Choose a *grading exponent* $\beta \geq 1$ and define a mesh on the interval $(0, 1)$ by

$$t_j = \begin{cases} \dfrac{1}{2}\left(\dfrac{2j}{n}\right)^{\beta} & \text{if } 0 \leq j \leq n/2, \\ 1 - t_{n-j} & \text{if } n/2 < j \leq n. \end{cases} \tag{2.6}$$

For $\beta = 1$ the mesh is uniform, but as β increases from 1, the points are more concentrated at each end of the interval. The length $\Delta t_j = t_j - t_{j-1}$ of the jth interval satisfies

$$\Delta t_j \simeq \frac{1}{n}\left(\frac{j}{n}\right)^{\beta-1} \simeq \Delta t_{n-j} \quad \text{for } 1 \leq j \leq n/2. \tag{2.7}$$

We construct the corresponding product mesh with n^2 elements on Γ with vertices

$$t_{(j_1,j_2)} = (t_{j_1}, t_{j_2}) \quad \text{for } 0 \leq j_1 \leq n \text{ and } 0 \leq j_2 \leq n. \tag{2.8}$$

Elements K near any corner are shape-regular with $h_K \simeq (1/n)^{\beta} \simeq \rho_K$. Away from the boundary they are also shape-regular with $h_K \simeq 1/n \simeq \rho_K$. However, near the middle of an edge we have $h_K \simeq 1/n$ and $\rho_K \simeq (1/n)^{\beta}$, so if $\beta > 1$ then degeneracy occurs with the maximum aspect ratio for the elements growing like $n^{\beta-1}$; see Figure 2.1. This construction can be generalised to other polyhedral surfaces, see, e.g. [11, 5] and Example 5.5 below.

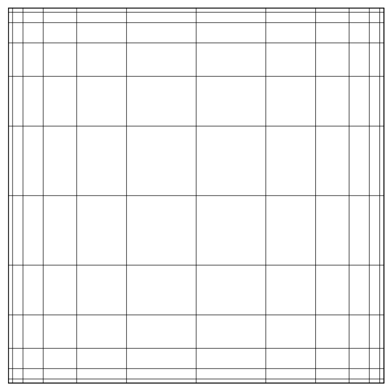

FIG. 2.1. *Power-graded tensor-product mesh with $\beta = 3$ and $N = 14^2$ elements.*

2.3. Condition number estimates. As an illustration of the results which we shall prove later in this paper, let us suppose we apply the Galerkin method to the weakly singular equation (2.1), with Γ given by (2.5) and with the subspace $X \subset \widetilde{H}^{-1/2}(\Gamma)$ chosen to be the space of piecewise-constant functions on the mesh (2.8). The dimension of X is $N = n^2$. In Theorems 5.7 and 7.4 we will prove that in this case the Galerkin matrix $\boldsymbol{B}$ satisfies the spectral bounds $\lambda_{\max}(\boldsymbol{B}) \lesssim N^{-1}$ and $\lambda_{\min}(\boldsymbol{B}) \gtrsim N^{-3\beta/2}$, whereas the diagonally-scaled Galerkin matrix $\boldsymbol{B}'$ satisfies

$$\lambda_{\max}(\boldsymbol{B}') \lesssim N^{1/2} \times \begin{cases} 1 & \text{if } 1 \leq \beta < 2, \\ (1+\log N)^{1/2} & \text{if } \beta = 2, \\ (1+\log N)^2 & \text{if } \beta > 2, \end{cases} \qquad \text{and} \qquad \lambda_{\min}(\boldsymbol{B}') \gtrsim \begin{cases} 1 & \text{if } \beta = 1, \\ (1+\log N)^{-1} & \text{if } \beta > 1. \end{cases}$$

Hence, $\operatorname{cond}(\boldsymbol{B})$ grows like $N^{(3\beta/2)-1}$ whereas $\operatorname{cond}(\boldsymbol{B}')$ essentially grows like $N^{1/2}$, which is the rate of growth in the case of shape-regular meshes.

On the other hand, suppose we solve (2.3), with Γ given by (2.5) and with $X \subset \widetilde{H}^{1/2}(\Gamma)$ chosen to be the space of continuous piecewise-bilinear functions on the mesh (2.8) which vanish at the boundary of Γ. The dimension of X is $N = (n-1)^2 = O(n^2)$. In Theorems 5.8 and 7.5, we shall prove that

$$\lambda_{\max}(\boldsymbol{B}) \lesssim N^{-1/2} \quad \text{and} \quad \lambda_{\min}(\boldsymbol{B}) \gtrsim \begin{cases} N^{-1}, & \text{for } 1 \leq \beta < 2, \\ N^{-1}(1+\log N)^{-1}, & \text{for } \beta = 2, \\ N^{-\beta/2}, & \text{for } \beta > 2, \end{cases}$$

whereas $\boldsymbol{B}'$ satisfies

$$\lambda_{\max}(\boldsymbol{B}') \lesssim 1 \quad \text{and} \quad \lambda_{\min}(\boldsymbol{B}') \gtrsim N^{-1/2} \times \begin{cases} 1, & \text{for } 1 \le \beta < 2, \\ (1+\log N)^{-1/2}, & \text{for } \beta = 2, \\ (1+\log N)^{-2}, & \text{for } \beta > 2. \end{cases}$$

Thus, in this case the condition number of $\boldsymbol{B}$ grows like $N^{(\beta-1)/2}$ (for $\beta > 2$), whereas the condition number of $\boldsymbol{B}'$ again essentially grows only like $N^{1/2}$ (for any $\beta \ge 1$), which is again the rate of growth in the shape-regular case.

3. General Framework. In this section we set up the theoretical apparatus in which the general spectral estimates of §5 will be proved. Recall that Γ denotes either a bounded (open or closed) d-dimensional Lipschitz surface in $\mathbb{R}^{d+1}$, for $d = 2$, or else a bounded Lipschitz domain in $\mathbb{R}^d$, for $d = 2$ or 3.

We define the Sobolev spaces $H^s(\Gamma)$, $\widetilde{H}^s(\Gamma)$, $|s| \le 1$ in the usual way; see, for example, [9] for details. In particular, when Γ is a Lipschitz domain or an open Lipschitz surface, $u \in \widetilde{H}^1(\Gamma)$ implies that u has vanishing trace on the boundary of Γ. For $0 < s < 1$, $\widetilde{H}^s(\Gamma)$ interpolates between $L_2(\Gamma)$ and $\widetilde{H}^1(\Gamma)$. In any case, $H^{-s}(\Gamma)$ is the dual of $\widetilde{H}^s(\Gamma)$ and $\widetilde{H}^{-s}(\Gamma)$ is the dual of $H^s(\Gamma)$, for all $|s| \le 1$. When Γ is a closed surface, $\widetilde{H}^s(\Gamma) = H^s(\Gamma)$ for all $|s| \le 1$. (Higher order Sobolev spaces can be defined on domains and on smooth enough surfaces, but we do not need these here.)

In what follows we will also be interested in Sobolev norms of various functions defined over subdomains $\hat{\Gamma} \subset \Gamma$. Since different equivalent norms for $H^s(\hat{\Gamma})$ or $\widetilde{H}^s(\hat{\Gamma})$ might scale differently with the size of $\hat{\Gamma}$, we follow the notation used in [1] and write $|||u|||_{H^s(\hat{\Gamma})}$ and $|||u|||_{\widetilde{H}^s(\hat{\Gamma})}$ to indicate the specific norms obtained for $|s| \le 1$ by real interpolation and duality, starting from the usual norm in $L_2(\hat{\Gamma})$ and the Sobolev norms

$$|||u|||^2_{H^1(\hat{\Gamma})} = \|u\|^2_{L_2(\hat{\Gamma})} + |u|^2_{H^1(\hat{\Gamma})} \quad \text{and} \quad |||u|||^2_{\widetilde{H}^1(\hat{\Gamma})} = |u|^2_{H^1(\hat{\Gamma})} = \sum_{|\alpha|=1} \|\partial^\alpha u\|^2_{L_2(\hat{\Gamma})} .$$

Note that $|\cdot|_{H^1(\hat{\Gamma})}$ is only a seminorm on $H^1(\hat{\Gamma})$ but is a norm on $\widetilde{H}^1(\hat{\Gamma})$. (The distinction between $\|\cdot\|_{H^s(\hat{\Gamma})}$ and $|||\cdot|||_{H^s(\hat{\Gamma})}$ is significant only when $\hat{\Gamma}$ is a proper subset of Γ. In what follows we will freely interchange $\|\cdot\|_{H^s(\Gamma)}$ and $|||\cdot|||_{H^s(\Gamma)}$.)

For later use, we recall [11, Lemma 3.2], [1, Theorem 4.1]: *if $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ is a partitioning of a bounded Lipschitz domain Γ into non-overlapping Lipschitz domains, then for $|s| \le 1$,*

$$|||v|||^2_{\widetilde{H}^s(\Gamma)} \le \sum_{j=1}^N |||v|||^2_{\widetilde{H}^s(\Gamma_j)} \quad \textit{and} \quad \sum_{j=1}^N |||u|||^2_{H^s(\Gamma_j)} \le |||u|||^2_{H^s(\Gamma)} . \tag{3.1}$$

We also note that (see [1, (4.1)])

$$\|u\|_{H^s(\Gamma)} \lesssim \|u\|_{\widetilde{H}^s(\Gamma)} \quad \text{if } u \in \widetilde{H}^s(\Gamma) \cap L_2(\Gamma), \text{ for all } |s| \le 1, \tag{3.2}$$

and that [9, p. 320]

$$|||v|||_{\widetilde{H}^s(\hat{\Gamma})} \lesssim |||v|||^{1-s}_{L_2(\hat{\Gamma})} |||v|||^s_{\widetilde{H}^1(\hat{\Gamma})} \quad \text{and} \quad |||v|||_{\widetilde{H}^{-s}(\hat{\Gamma})} \lesssim |||v|||^{1-s}_{L_2(\hat{\Gamma})} |||v|||^s_{\widetilde{H}^{-1}(\hat{\Gamma})} \quad \text{for } 0 < s < 1. \tag{3.3}$$

As mentioned in §1, we will be considering a family of partitions $\{\mathcal{P}\}$ of Γ. Each partition $\mathcal{P}$ consists of relatively open, pairwise-disjoint finite elements $K \subset \Gamma$ with the property $\overline{\Gamma} = \cup\{\overline{K} : K \in \mathcal{P}\}$. For each $K \in \mathcal{P}$, h_K denotes its diameter and ρ_K the diameter of the largest sphere whose intersection with $\overline{\Gamma}$ lies entirely inside $\overline{K}$. Also, for any measurable subset S of Γ, $|S|$ denotes its d-dimensional measure.

In order to impose a simple geometric character on the mesh $\mathcal{P}$, we assume that each $K \in \mathcal{P}$ is diffeomorphic to a simple reference element. More precisely, let $\hat{\sigma}^d$ denote the unit simplex and $\hat{\kappa}^d = [0,1]^d$ the unit cube in $\mathbb{R}^d$. Thus, $\hat{\sigma}^2$ is the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ and $\hat{\sigma}^3$ is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

We assume that for each $K \in \mathcal{P}$, there exists a reference element $\hat{K} = \hat{\sigma}^d$ or $\hat{\kappa}^d$ and a bijective map $\chi_K : \hat{K} \to K$, with both χ_K and χ_K^{-1} smooth. (Here, for simplicity, "smooth" means $\mathcal{C}^\infty$.) Each element has vertices and edges, defined to be the images of the vertices and edges of the corresponding unit element under χ_K. In the $3D$ case, the element also has faces, comprising the images of the faces of the unit element. We assume each partition is *conforming*, i.e., for each K, $K' \in \mathcal{P}$ with $K \neq K'$, the intersection $\overline{K} \cap \overline{K'}$ is allowed to be either empty, a vertex, an edge or (when $d = 3$) a face of both K and K'. The requirement that χ_K is smooth ensures that edges of Γ ($d = 2$) and edges of $\partial\Gamma$ ($d = 3$) are confined to edges of elements $K \in \mathcal{P}$.

Let J_K denote the $3 \times d$ Jacobian of χ_K. Then

$$\int_K f(\mathbf{x}) dx = \int_{\hat{K}} f(\chi_K(\hat{\mathbf{x}})) g_K(\hat{\mathbf{x}}) d\hat{x} \quad \text{where} \quad g_K := \left(\det J_K^T J_K\right)^{1/2} . \tag{3.4}$$

In addition to the assumption that χ_K and χ_K^{-1} are smooth, we also require the following assumption on J_K:

Assumption 3.1. *There exist positive constants D, E such that*

$$D^{-1}|K|^2 \leq \det\left(J_K(\hat{\mathbf{x}})^T J_K(\hat{\mathbf{x}})\right) \leq D|K|^2, \tag{3.5a}$$

$$E\rho_K^2 \leq \lambda_{\min}\left(J_K(\hat{\mathbf{x}})^T J_K(\hat{\mathbf{x}})\right), \tag{3.5b}$$

uniformly for $\hat{\mathbf{x}} \in \hat{K}$, $K \in \mathcal{P}$, and $\mathcal{P} \in \mathcal{F}$.

Assumption 3.1 holds, for example, when K is a planar triangle ($d = 2$) or a tetrahedron ($d = 3$) and χ_K is affine. It is also satisfied by bilinear maps from the unit square to quadrilaterals ($d = 2$), provided the quadrilaterals are not too far from parallelograms. These and other examples are explored in [7].

Assumption 3.1 describes the quality of the maps which take the unit element $\hat{K}$ to each K. We also need assumptions on how the size and shape of neighbouring elements in our mesh may vary. Here we impose *only very weak local conditions which require the meshes to be neither quasi-uniform nor shape-regular.* In addition, we need a uniform bound on the number of elements that touch the ith node $\mathbf{x}_i$.

Assumption 3.2. *There exist positive constants F, G, H and an integer M such that for all $\mathcal{P} \in \mathcal{F}$,*

$$h_K \leq F\ h_{K'}\,, \quad \rho_K \leq G\ \rho_{K'}\,, \quad |K| \leq H\ |K'|\,, \qquad \text{for all } K, K' \in \mathcal{P} \text{ with } \overline{K} \cap \overline{K'} \neq \emptyset, \tag{3.6a}$$

$$\text{and also} \qquad \max_{i \in \mathcal{N}} \#\{K \in \mathcal{P} : \mathbf{x}_i \in \overline{K}\} \ \leq\ M\,. \tag{3.6b}$$

Note that condition (3.6a) requires that h_K and ρ_K do not vary too rapidly between neighbouring elements. This allows elements with large aspect ratio, provided their immediate neighbours have a comparable aspect ratio. It is clear that the power meshes (2.8) satisfy Assumption 3.2.

From now on, if $A(\mathcal{P})$ and $B(\mathcal{P})$ are two mesh-dependent quantities, then the inequality $A(\mathcal{P}) \lesssim B(\mathcal{P})$ will mean that there is a constant C independent of $\mathcal{P}$, such that $A(\mathcal{P}) \leq CB(\mathcal{P})$. ($C$ may depend on D, E, F, G or M.) Also the notation $A(\mathcal{P}) \simeq B(\mathcal{P})$ will mean that $A(\mathcal{P}) \lesssim B(\mathcal{P})$ and $B(\mathcal{P}) \lesssim A(\mathcal{P})$.

For an integer $\ell \geq 0$ and a reference element $\hat{K} \in \{\hat{\sigma}^d, \hat{\kappa}^d\}$, we define

$$\mathbb{P}^\ell(\hat{K}) = \begin{cases} \text{polynomials of total degree } \leq \ell \text{ on } \hat{K} & \text{if } \hat{K} = \hat{\sigma}^d, \\ \text{polynomials of coordinate degree } \leq \ell \text{ on } \hat{K} & \text{if } \hat{K} = \hat{\kappa}^d, \end{cases}$$

and the finite element spaces

$$\mathcal{S}_0^\ell(\mathcal{P}) = \{u \in L^\infty(\Gamma) : u \circ \chi_K \in \mathbb{P}^\ell(\hat{K}), K \in \mathcal{P}\} \quad \text{for } \ell \geq 0,$$
$$\mathcal{S}_1^\ell(\mathcal{P}) = \{u \in C^0(\Gamma) : u \circ \chi_K \in \mathbb{P}^\ell(\hat{K}), K \in \mathcal{P}\} \quad \text{for } \ell \geq 1.$$

Finally we introduce suitable bases for these spaces. In this paper we consider standard nodal bases defined as follows. Let $d(\ell)$ denote the dimension of $\mathbb{P}^\ell(\hat{K})$ and choose a set of nodes $\{\hat{\mathbf{x}}_p : p = 1, \ldots, d(\ell)\} \subset \overline{\hat{K}}$ with the property that each $\hat{u} \in \hat{\mathbb{P}}^\ell(\hat{K})$ is uniquely determined by its values at the $\hat{\mathbf{x}}_p$. Then there are basis functions $\{\hat{\phi}_p\ , p = 1, \ldots, d(\ell)\}$ with the property $\hat{\phi}_p(\hat{\mathbf{x}}_q) = \delta_{p,q}$. From these we can define basis functions on the open set K (implicitly) by $\phi_{p,K} \circ \chi_K = \hat{\phi}_p$. We extend $\phi_{p,K}$ to $\overline{K}$ by continuity and then (discontinuously) to the whole of Γ by zero. If we introduce the nodes $\mathbf{x}_{p,K} := \chi_K(\hat{\mathbf{x}}_p) \in \overline{K}$ then clearly

$$\phi_{p,K}(\mathbf{x}_{q,K'}) = \delta_{(p,K),(q,K')}, \quad \text{for } \ p, q = 1, \ldots, d(\ell), \ \ K, K' \in \mathcal{P}\ . \tag{3.7}$$

The functions

$$\{\phi_{p,K} :\ p = 1, \ldots d(\ell),\ K \in \mathcal{P}\} \tag{3.8}$$

then constitute a suitable basis of $\mathcal{S}_0^m(\mathcal{P})$. When $\ell = 0$ we have the simple piecewise constant functions, and the nodes $\mathbf{x}_K = \mathbf{x}_{1,K}$ can be chosen as the centroids of each K.

For $\mathcal{S}_1^\ell(\mathcal{P})$, we require further that if two elements K and K' share a common edge e, then this edge is parametrised *equally from both sides.* More precisely, we require that if $\chi_K^{-1}(e) = \hat{e}$ and $\chi_{K'}^{-1}(e) = \hat{e}'$ then there exists an affine mapping $\gamma : \hat{e} \to \hat{e}'$ such that χ_K and $\chi_{K'} \circ \gamma$ coincide pointwise on $\hat{e}$. We assume that the points $\mathbf{x}_{p,K}$ and $\mathbf{x}_{p,K'}$ restricted to e coincide and that the values of u at these points are sufficient to determine uniquely $u|_e$ on e. (This condition is satisfied in the simplest case when χ_K and $\chi_{K'}$ are both affine maps.) In this case any $u \in \mathcal{S}_1^\ell(\mathcal{P})$ is determined uniquely by its values at the set of global nodes $\{\mathbf{x}_{p,K} : \ \ p = 1, \ldots, d(\ell), \ K \in \mathcal{P}\}$, where coincident nodes on the boundary of more than one element now constitute a single degree of freedom. Denoting this set more abstractly by $\{\mathbf{x}_k : k \in \mathcal{N}\}$ for some suitable index set of nodes (or degrees of freedom) $\mathcal{N}$, our basis for $\mathcal{S}_1^\ell(\mathcal{P})$ is

$$\{\phi_k : k \in \mathcal{N}\} \ , \tag{3.9}$$

where $\phi_k \in \mathcal{S}_1^\ell(\mathcal{P})$ is the unique function satisfying

$$\phi_k(\mathbf{x}_{k'}) = \delta_{k,k'} \quad \text{for all } k, \, k' \in \mathcal{N}. \tag{3.10}$$

A simple example is the space of the continuous bilinear functions on a mesh of quadilaterals, with nodes chosen to be the vertices of the elements.

Clearly the basis (3.8) may be written in the abstract form (3.9) by allowing the set $\mathcal{N}$ to contain double indices of the form (p, K). With this notation, (3.10) follows from (3.7). Moreover, in any case,

$$\Gamma_k \subseteq \bigcup \left\{\overline{K} : \mathbf{x}_k \in \overline{K}\right\} \ , \quad \text{where } \Gamma_k := \operatorname{supp} \phi_k. \tag{3.11}$$

Throughout the rest of the paper N denotes the cardinality of the nodal set $\mathcal{N}$.

4. Sobolev Norm of a Nodal Basis Function. We now establish some technical estimates needed in the next section in our proofs of the spectral bounds for $\boldsymbol{B}$ and $\boldsymbol{B'}$. For these we need the following notation. For $k \in \mathcal{N}$, define

$$h_k = \text{average of those } h_K \text{ for which } x_k \in \overline{K}, \qquad \rho_k = \text{average of those } \rho_K \text{ for which } x_k \in \overline{K},$$

and note that the second inequality in (3.6a) implies that

$$\min_{x_k \in \overline{K}} \rho_K \lesssim \rho_k \lesssim \max_{x_k \in \overline{K}} \rho_K \quad \text{for } k \in \mathcal{N}. \tag{4.1}$$

The following Theorem is closely related to [7, Theorems 3.2 and 3.6].

THEOREM 4.1. *Let $\{\phi_k\}_{k\in\mathcal{N}}$ be a nodal basis for $\mathcal{S}_i^\ell(\mathcal{P}) \subset \widetilde{H}^m(\Gamma)$, where $i = 0$ or 1.*

(i) If $0 \le m \le 1$, *then* $\quad |||\phi_k|||_{\widetilde{H}^m(\Gamma_k)} \lesssim \rho_k^{-m} \|\phi_k\|_{L_2(\Gamma_k)}$.

(ii) If $-1 \le m \le 0$ *then* $\quad \rho_k^{-m} \|\phi_k\|_{L_2(\Gamma_k)} \lesssim |||\phi_k|||_{H^m(\Gamma_k)}$.

Proof. The proof follows the same lines as the proofs of [7, Theorems 3.2 and 3.6], in which the same result is proved on the whole domain Γ. To get the proof of the present result, one only has to check that the arguments in [7] remain true if the global norm $|||\cdot|||_{\widetilde{H}^s(\Gamma)}$ is replaced by the local norm $|||\cdot|||_{\widetilde{H}^s(\Gamma_k)}$. We recall that the latter norm is obtained for $s \in (0,1)$ by interpolation between the norms $\|\cdot\|_{L_2(\Gamma_k)}$ and $|\cdot|_{H^1(\Gamma_k)}$ and then by duality for $s \in [-1, 0]$. Now, following the arguments in [7, Theorem 3.2], it is easily seen that

$$|\phi_k|^2_{H^1(\Gamma_k)} = \sum_{\substack{K\in\mathcal{P}\\ K\subset\Gamma_k}} \int_K |\nabla \phi_k|^2 \lesssim \rho_k^{-2} \sum_{\substack{K\in\mathcal{P}\\ K\subset\Gamma_k}} \|\phi_k\|^2_{L_2(K)} = \rho_k^{-2} \|\phi_k\|^2_{L_2(\Gamma_k)}.$$

The proof of (i) for $m = 1$ follows directly and result (i) then follows by interpolation (3.3) .

For (ii), note first that when $m \in [-1, 0]$, the definition of the dual space implies

$$|||\phi_k|||_{H^m(\Gamma_k)} \ge \frac{|(\phi_k, w)_{\Gamma_k}|}{|||w|||_{\widetilde{H}^{-m}(\Gamma_k)}} \ , \tag{4.2}$$

for any $w \in \widetilde{H}^{-m}(\Gamma_k)$, not identically zero. The proof is completed by constructing a test function $w \in \widetilde{H}^{-m}(\Gamma_k)$ with the properties

$$|(\phi_k, w)_{\Gamma_k}| \gtrsim \rho_k^2 \|\phi_k\|^2_{L_2(\Gamma_k)} , \tag{4.3}$$

$$|||w|||_{\widetilde{H}^{-m}(\Gamma_k)} \lesssim \rho_k^{2+m} \|\phi_k\|_{L_2(\Gamma_k)} . \tag{4.4}$$

The required construction of w is given in the proof of [7, Theorem 3.6], where the estimates (4.3) and (4.4) with $m = -1$ are established (see [7, eqns (3.14), (3.15)] and put $\alpha = 0$ and $k = 1$). The proof of (4.4) for $m \in [-1, 0]$ is obtained by establishing it for $m = 0$ and then interpolating with $m = -1$. To establish (4.4) for $m = 0$ one has to look closely at the argument in [7, Thm 3.6]. For any $K \in \mathcal{P}$, $K \subset \Gamma_k$, it is shown that there exists a subset $t(K) \subset K$ and a function $P_{t(K)} \in \widetilde{H}^1(K)$ such that $\phi_k|_{t(K)}$ is one-signed and such that $\|P_{t(K)}\|_{L_2(K)} \simeq |t(K)|^{1/2}$. The hidden constants in this estimate are independent of k, K and the mesh. Then the w which satisfies (4.3) and (4.4) with $m = -1$ is defined as $w = \sum_{K \subset \Gamma_k} \rho_K^2 \mathrm{sign}(\phi_k|_{t(K)}) \inf_{x \in t(K)} |\phi_k(x)| P_{t(K)}$. Then

$$\|w\|^2_{L_2(\Gamma_k)} = \sum_{K \subset \Gamma_k} \rho_K^4 \Big(\inf_{x \in t(K)} |\phi_k(x)| \Big)^2 \|P_{t(K)}\|^2_{L_2(t(K))} \lesssim \rho_k^4 \sum_{K \subset \Gamma_k} \Big(\inf_{x \in t(K)} |\phi_k(x)| \Big)^2 |t(K)| \lesssim \rho_k^4 \|\phi_k\|^2_{L_2(\Gamma_k)} ,$$

as required. □

Theorem 4.1 is the key component in the proof of the next result, which in turn is a partial generalisation of [1, Lemma 4.7] and [1, Theorem 4.8].

Theorem 4.2. *Let $k \in \mathcal{N}$.*

(i) If $1 \le p \le \infty$ then $\|\phi_k\|_{L_p(\Gamma)} = \|\phi_k\|_{L_p(\Gamma_k)} \simeq |\Gamma_k|^{1/p}$.

(ii) If $0 \le m \le 1$ then $\|\phi_k\|^2_{\widetilde{H}^m(\Gamma)} \lesssim |||\phi_k|||^2_{\widetilde{H}^m(\Gamma_k)} \lesssim |\Gamma_k| \rho_k^{-2m}$.

(iii) If $-1 \le m \le 0$ then $|\Gamma_k| \rho_k^{-2m} \lesssim |||\phi_k|||^2_{H^m(\Gamma_k)} \lesssim \|\phi_k\|^2_{H^m(\Gamma)}$.

(iv) If $0 \le 2m < d$ then $\|\phi_k\|^2_{H^m(\Gamma)} \gtrsim |\Gamma_k|^{1-2m/d}$.

(v) If $-d < 2m \le 0$ then $\|\phi_k\|^2_{\widetilde{H}^m(\Gamma)} \lesssim |\Gamma_k|^{1-2m/d}$.

Proof. (i) By the definition of ϕ_k,

$$\|\phi_k\|^p_{L_p(\Gamma_k)} = \sum_{\substack{K \in \mathcal{P} \\ K \subset \Gamma_k}} \int_K |\phi_k|^p.$$

For a typical $K \subset \Gamma_k$, recall (3.4) and write $\int_K |\phi_k|^p = \int_{\hat{K}} |\hat{\phi}_k|^p g_K \simeq |K| \int_{\hat{K}} |\hat{\phi}_k|^p \simeq |K|$, by Assumption 3.1. Now sum over all elements $K \subset \Gamma_k$ to obtain the result.

The left-hand inequality in (ii) follows directly from the left-hand inequality in (3.1), while the right-hand inequality in (ii) follows from part (i) of Theorem 4.1 and part (i) of the present theorem.

Similarly, in part (iii), we use the right-hand inequality in (3.1) and part (ii) of Theorem 4.1, combined with part (i) of the present theorem.

To prove (iv), we put $p = 2d/(d-2m) \in [2, \infty)$ and apply the Sobolev imbedding theorem together with part (i) above to obtain

$$\|\phi_k\|^2_{H^m(\Gamma)} \gtrsim \|\phi_k\|^2_{L_p(\Gamma)} \simeq |\Gamma_k|^{2/p} = |\Gamma_k|^{1-2m/d}.$$

Part (v) follows using a dual imbedding: for $q = 2d/(d-2m) \in (1, 2]$,

$$\|\phi_k\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \|\phi_k\|^2_{L_q(\Gamma)} \simeq |\Gamma_k|^{2/q} = |\Gamma_k|^{1-2m/d}.$$

□

5. Bounds on the Extremal Eigenvalues. In this section we obtain general bounds on the spectra of the matrices $\boldsymbol{B}$ and $\boldsymbol{B}'$ which were defined in (1.4) and (1.6). Since $\boldsymbol{B}$ is symmetric, these may be obtained by estimating the Rayleigh quotient $\boldsymbol{\alpha}^T \boldsymbol{B} \boldsymbol{\alpha} / \boldsymbol{\alpha}^T \boldsymbol{\alpha}$ from above and from below. For a typical $v \in X$ we write

$$v = \sum_{k \in \mathcal{N}} v_k \quad \text{where } v_k = \alpha_k \phi_k \text{ and } \alpha_k = v(x_k). \tag{5.1}$$

Then, since

$$\boldsymbol{\alpha}^T \boldsymbol{B} \boldsymbol{\alpha} = B(v,v) \simeq \|v\|^2_{\widetilde{H}^m(\Gamma)} \quad \text{and} \quad \boldsymbol{\alpha}^T \boldsymbol{\alpha} = \sum_{k\in\mathcal{N}} v(x_k)^2,$$

if we show the bounds

$$\lambda_X \sum_{k\in\mathcal{N}} v(x_k)^2 \lesssim \|v\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \Lambda_X \sum_{k\in\mathcal{N}} v(x_k)^2 \quad \text{for all } v \in X, \tag{5.2}$$

then we have the estimates: $\lambda_{\max}(\boldsymbol{B}) \lesssim \Lambda_X$ and $\lambda_{\min}(\boldsymbol{B}) \gtrsim \lambda_X$.

Similarly, for the diagonally-scaled matrix $\boldsymbol{B}'$, note that

$$\boldsymbol{\alpha}^T \boldsymbol{D} \boldsymbol{\alpha} = \sum_{k\in\mathcal{N}} \alpha_k^2 B(\phi_k,\phi_k) = \sum_{k\in\mathcal{N}} B(v_k,v_k) \simeq \sum_{k\in\mathcal{N}} \|v_k\|^2_{\widetilde{H}^m(\Gamma)}.$$

Thus if we can show that

$$\lambda'_X \sum_{k\in\mathcal{N}} \|v_k\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \|v\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \Lambda'_X \sum_{k\in\mathcal{N}} \|v_k\|^2_{\widetilde{H}^m(\Gamma)} \quad \text{for all } v \in X, \tag{5.3}$$

then it will follow that $\lambda_{\max}(\boldsymbol{B}') \lesssim \Lambda'_X$ and $\lambda_{\min}(\boldsymbol{B}') \gtrsim \lambda'_X$.

For each element $K \in \mathcal{P}$, let $\mathcal{N}(K) = \{\, k \in \mathcal{N} : \Gamma_k \cap K \neq \emptyset \,\}$. Our assumptions on the family of partitions $\{\mathcal{P}\}$ imply that

$$\text{the cardinality of } \mathcal{N}(K) \text{ for } K \in \mathcal{P} \text{ is bounded independently of } \mathcal{P} \tag{5.4}$$

and that for each $\mathcal{P}$ the index set $\mathcal{N}$ may be partitioned into disjoint subsets $\mathcal{N}_1$, $\mathcal{N}_2$, …, $\mathcal{N}_L$ having the property

$$\text{interior}(\operatorname{supp}\phi_k) \cap \text{interior}(\operatorname{supp}\phi_{k'}) = \emptyset \quad \text{if } k,\ k' \in \mathcal{N}_\ell \text{ and } k \neq k', \tag{5.5}$$

in such a way that L is bounded independently of $\mathcal{P}$.

In Lemmas 5.1 and 5.2 we will obtain bounds on the spectra of $\boldsymbol{B}$ and $\boldsymbol{B}'$, some of which involve the quantities:

$$\Phi_{m,k} := \frac{|\Gamma_k| \rho_k^{-2m}}{\|\phi_k\|^2_{\widetilde{H}^m(\Gamma)}}\ , \quad k \in \mathcal{N}\ . \tag{5.6}$$

Simple bounds on $\Phi_{m,k}$ may be obtained by employing Theorem 4.2 and (3.2) to obtain:

$$\Phi_{m,k} \lesssim 1 \text{ for } -1 \le m \le 0 \qquad \text{and} \qquad \Phi_{m,k} \gtrsim \frac{|\Gamma_k|^{2m/d}}{\rho_k^{2m}} \text{ for } -d < 2m \le 0. \tag{5.7}$$

$$\Phi_{m,k} \gtrsim 1 \text{ for } 0 \le m \le 1 \qquad \text{and} \qquad \Phi_{m,k} \lesssim \frac{|\Gamma_k|^{2m/d}}{\rho_k^{2m}} \text{ for } 0 \le 2m < d, \tag{5.8}$$

(Note that in the shape-regular case, (5.8) and (5.7) are sharp estimates, since $|\Gamma_k|^{1/d} \simeq h_k \simeq \rho_k$, and so $\Phi_{m,k} \simeq 1$ for $2|m| < d$.)

In the next two lemmas we shall decompose an arbitrary $v \in X \subset \widetilde{H}^m(\Gamma)$ as in (5.1).

Lemma 5.1. *For $-1 \le m \le 0$ and $-d < 2m$, we have*

$$\min_{k\in\mathcal{N}} |\Gamma_k| \rho_k^{-2m} \ \lesssim\ \lambda_{\min}(\boldsymbol{B}) \ \le\ \lambda_{\max}(\boldsymbol{B}) \ \lesssim\ \left(\sum_{k\in\mathcal{N}} |\Gamma_k|^{1-d/2m} \right)^{-2m/d},$$

$$\min_{k\in\mathcal{N}} \Phi_{m,k} \ \lesssim\ \lambda_{\min}(\boldsymbol{B}') \ \le\ \lambda_{\max}(\boldsymbol{B}') \ \lesssim\ \left(\sum_{k\in\mathcal{N}} |\Gamma_k| \rho_k^{-d} \right)^{-2m/d}.$$

The lower bounds continue to hold if the hypothesis is weakened to just $-1 \le m \le 0$.

Proof. With $q = 2d/(d-2m) \in (1,2]$, and using the dual Sobolev embedding, we obtain $\|v\|_{\widetilde{H}^m(\Gamma)} \lesssim \|v\|_{L_q(\Gamma)}$. Now, using Hölder's inequality and property (5.5), we obtain

$$\|v\|_{\widetilde{H}^m(\Gamma)} \lesssim \|v\|_{L_q(\Gamma)} \lesssim \sum_{\ell=1}^{L} \left\| \sum_{k\in\mathcal{N}_\ell} v_k \right\|_{L_q(\Gamma)} \lesssim \left(\sum_{\ell=1}^{L} \left\| \sum_{k\in\mathcal{N}_\ell} v_k \right\|_{L_q(\Gamma)}^q \right)^{1/q} \lesssim \left(\sum_{k\in\mathcal{N}} \|v_k\|_{L_q(\Gamma)}^q \right)^{1/q} . \tag{5.9}$$

Moreover from Hölder's inequality with $1/p + q/2 = 1$ and with w_k denoting any positive weight, we have

$$\sum_{k\in\mathcal{N}} \|v_k\|_{L_q(\Gamma)}^q \le \left(\sum_{k\in\mathcal{N}} (w_k^{q/2})^p \right)^{1/p} \left(\sum_{k\in\mathcal{N}} (w_k^{-q/2} \|v_k\|_{L_q(\Gamma)}^q)^{2/q} \right)^{q/2} \lesssim \left(\sum_{k\in\mathcal{N}} w_k^{pq/2} \right)^{1/p} \left(\sum_{k\in\mathcal{N}} w_k^{-1} \|v_k\|_{L_q(\Gamma)}^2 \right)^{q/2} .$$

Combining this with (5.9), we obtain

$$\|v\|_{\widetilde{H}^m(\Gamma)}^2 \lesssim \left(\sum_{k\in\mathcal{N}} w_k^{pq/2} \right)^{2/(pq)} \sum_{k\in\mathcal{N}} w_k^{-1} \|v_k\|_{L_q(\Gamma)}^2 . \tag{5.10}$$

Note that $2/q = 1 - 2m/d,\ \ p = 1 - d/(2m)$ and $pq/2 = -d/(2m)$. By choosing $w_k = |\Gamma_k|^{2/q}$ in (5.10) and applying Theorem 4.2 (i), we obtain

$$\|v\|_{\widetilde{H}^m(\Gamma)}^2 \lesssim \left(\sum_{k\in\mathcal{N}} |\Gamma_k|^{1-d/2m} \right)^{-2m/d} \sum_{k\in\mathcal{N}} v(x_k)^2,$$

which, together with (5.2) implies the upper bound for $\lambda_{\max}(\boldsymbol{B})$.

On the other hand, with $w_k = |\Gamma_k|^{2/q-1}\rho_k^{2m}$, it follows from Theorem 4.2 (i), (iii) that

$$w_k^{-1} \|v_k\|_{L_q(\Gamma)}^2 \simeq v(x_k)^2 |\Gamma_k| \rho_k^{-2m} \lesssim v(x_k)^2 \|\phi_k\|_{H^m(\Gamma)}^2 = \|v_k\|_{H^m(\Gamma)}^2 .$$

Since $w_k^{pq/2} = |\Gamma_k|^{p-pq/2} (\rho_k^{2m})^{pq/2} = |\Gamma_k| \rho_k^{-d}$, (5.10) leads to

$$\|v\|_{\widetilde{H}^m(\Gamma)}^2 \lesssim \left(\sum_{k\in\mathcal{N}} |\Gamma_k| \rho_k^{-d} \right)^{-2m/d} \sum_{k\in\mathcal{N}} \|v_k\|_{H^m(\Gamma)}^2 .$$

The upper bound for $\lambda_{\max}(\boldsymbol{B}')$ follows by (3.2) and (5.3).

Now consider the lower bounds. Given $k \in \mathcal{N}$, choose an element $K \in \mathcal{P}$ such that $x_k \in \overline{K}$. Then, with $\hat{v} = v \circ \chi_K$, and using equivalence of norms on finite dimensional spaces, we have

$$v(x_k)^2 \le \|v\|_{L_\infty(K)}^2 = \|\hat{v}\|_{L_\infty(\hat{K})}^2 \simeq \|\hat{v}\|_{L_2(\hat{K})}^2 \simeq |K|^{-1} \|v\|_{L_2(K)}^2 . \tag{5.11}$$

Moreover using [7, Theorem 3.6, Remark 3.8] applied on the single element K, we obtain $\|v\|_{L_2(K)} \lesssim \rho_K^m |||v|||_{H^m(K)}$. Combining this with (5.11) and using Assumption 3.2, we get

$$v(x_k)^2 \lesssim |K|^{-1} \rho_K^{2m} |||v|||_{H^m(K)}^2 \lesssim |\Gamma_k|^{-1} \rho_k^{2m} |||v|||_{H^m(K)}^2 . \tag{5.12}$$

Hence using (5.4) and (3.1),

$$\sum_{k\in\mathcal{N}} v(x_k)^2 \lesssim \left(\max_{k\in\mathcal{N}} |\Gamma_k|^{-1} \rho_k^{2m} \right) \sum_{K\in\mathcal{P}} |||v|||_{H^m(K)}^2 \lesssim \left(\max_{k\in\mathcal{N}} |\Gamma_k|^{-1} \rho_k^{2m} \right) \|v\|_{H^m(\Gamma)}^2,$$

and the lower bound for $\lambda_{\min}(\boldsymbol{B})$ follows by (3.2). To obtain the lower bound for $\lambda_{\min}(\boldsymbol{B}')$, we use the definition (5.6) of $\Phi_{m,k}$ combined with (5.12) to obtain

$$\|v_k\|_{\widetilde{H}^m(\Gamma)}^2 = v(x_k)^2 \|\phi_k\|_{\widetilde{H}^m(\Gamma)}^2 = \Phi_{m,k}^{-1} \left[v(x_k)^2 |\Gamma_k| \rho_k^{-2m} \right] \lesssim \Phi_{m,k}^{-1} |||v|||_{H^m(K)}^2 . \tag{5.13}$$

Then the required estimate follows by summing over k and using (3.1). ☐

LEMMA 5.2. *For $0 \le m \le 1$ and $2m < d$, we have*

$$\left(\sum_{k\in\mathcal{N}} |\Gamma_k|^{1-d/2m}\right)^{-2m/d} \lesssim \lambda_{\min}(\boldsymbol{B}) \le \lambda_{\max}(\boldsymbol{B}) \lesssim \max_{k\in\mathcal{N}} |\Gamma_k|\rho_k^{-2m} ,$$
$$\left(\sum_{k\in\mathcal{N}} |\Gamma_k|\rho_k^{-d}\right)^{-2m/d} \lesssim \lambda_{\min}(\boldsymbol{B}') \le \lambda_{\max}(\boldsymbol{B}') \lesssim \max_{k\in\mathcal{N}} \Phi_{m,k} .$$

The upper bounds continue to hold if the hypothesis is weakened to just $0 \le m \le 1$.

Proof. Using the decomposition (5.1) of v, we have

$$\|v\|^2_{\widetilde{H}^m(\Gamma)} = \left\|\sum_{\ell=1}^{L}\sum_{k\in\mathcal{N}_\ell} v_k\right\|^2_{\widetilde{H}^m(\Gamma)} \le \left(\sum_{\ell=1}^{L}\left\|\sum_{k\in\mathcal{N}_\ell} v_k\right\|_{\widetilde{H}^m(\Gamma)}\right)^2 \le L\sum_{\ell=1}^{L}\left\|\sum_{k\in\mathcal{N}_\ell} v_k\right\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \sum_{k\in\mathcal{N}} |||v_k|||^2_{\widetilde{H}^m(\Gamma_k)} , \tag{5.14}$$

where we used the left-hand inequality in (3.1) and the property (5.5). By Theorem 4.2 (ii),

$$|||v_k|||^2_{\widetilde{H}^m(\Gamma_k)} = v(x_k)^2|||\phi_k|||^2_{\widetilde{H}^m(\Gamma_k)} \lesssim v(x_k)^2|\Gamma_k|\rho_k^{-2m} . \tag{5.15}$$

Substituting this into (5.14) yields

$$\|v\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \left(\max_{j\in\mathcal{N}} |\Gamma_j|\rho_j^{-2m}\right)\sum_{k\in\mathcal{N}} v(x_k)^2,$$

which, recalling (5.2), implies the upper bound for $\lambda_{\max}(\boldsymbol{B})$.

To obtain the upper bound for $\lambda_{\max}(\boldsymbol{B}')$, we use (5.15) to write

$$|||v_k|||^2_{\widetilde{H}^m(\Gamma_k)} \lesssim v(x_k)^2|\Gamma_k|\rho_k^{-2m} = v(x_k)^2\Phi_{m,k}\|\phi_k\|^2_{\widetilde{H}^m(\Gamma)} = \Phi_{m,k}\|v_k\|^2_{\widetilde{H}^m(\Gamma)} .$$

Then we combine this with (5.14) and (5.3) to obtain the result.

Now we consider the lower bounds. Let $p = 2d/(d-2m) \in [2,\infty)$ so that $\|v\|_{L_p(\Gamma)} \lesssim \|v\|_{H^m(\Gamma)}$. Clearly $v(x_k)^2 \le \|v\|^2_{L_\infty(K)}$, for some $K \in \mathcal{P}$, $K \subseteq \Gamma_k$. Also $v \circ \chi_K = \hat{v}$, with $\hat{v} \in \mathbb{P}^\ell(\hat{K})$ and by equivalence of norms on finite dimensional spaces, combined with Assumptions 3.1 and 3.2, we have

$$v(x_k)^2 \le \|v\|^2_{L_\infty(K)} = \|\hat{v}\|^2_{L_\infty(\hat{K})} \simeq \|\hat{v}\|^2_{L_p(\hat{K})} \simeq |K|^{-2/p}\|v\|^2_{L_p(K)} \lesssim |\Gamma_k|^{-2/p}\|v\|^2_{L_p(\Gamma_k)} . \tag{5.16}$$

Hence, by Hölder's inequality with $2/p + 1/q = 1$,

$$\sum_{k\in\mathcal{N}} v(x_k)^2 \lesssim \left(\sum_{j\in\mathcal{N}} \left(|\Gamma_j|^{-2/p}\right)^q\right)^{1/q}\left(\sum_{k\in\mathcal{N}} \|v\|^p_{L_p(\Gamma_k)}\right)^{2/p} \lesssim \left(\sum_{j\in\mathcal{N}} |\Gamma_j|^{-2q/p}\right)^{1/q}\|v\|^2_{L_p(\Gamma)} ,$$

where we used the property (5.4). Now, since $2/p = 1 - 2m/d$ we have $1/q = 2m/d$ and

$$\sum_{k\in\mathcal{N}} v(x_k)^2 \lesssim \left(\sum_{j\in\mathcal{N}} |\Gamma_j|^{1-d/2m}\right)^{2m/d}\|v\|^2_{H^m(\Gamma)},$$

which, in view of (3.2) and (5.2), proves the lower bound for $\lambda_{\min}(\boldsymbol{B})$.

To estimate $\lambda_{\min}(\boldsymbol{B}')$, we use Theorem 4.2 (ii) and (5.16) to obtain

$$\|v_k\|^2_{\widetilde{H}^m(\Gamma)} = v(x_k)^2\|\phi_k\|^2_{\widetilde{H}^m(\Gamma)} \lesssim v(x_k)^2|\Gamma_k|\rho_k^{-2m} \lesssim |\Gamma_k|^{1-2/p}\rho_k^{-2m}\|v\|^2_{L_p(\Gamma_k)} . \tag{5.17}$$

Thus, recalling $1 - 2/p = 2m/d$ and employing again (5.4),

$$\sum_{k\in\mathcal{N}} \|v_k\|^2_{\widetilde{H}^m(\Gamma)} \lesssim \sum_{k\in\mathcal{N}} |\Gamma_k|^{2m/d}\rho_k^{-2m}\|v\|^2_{L_p(\Gamma_k)}$$
$$\lesssim \left(\sum_{j\in\mathcal{N}} \left(|\Gamma_j|^{2m/d}\rho_j^{-2m}\right)^q\right)^{1/q}\left(\sum_{k\in\mathcal{N}} \|v\|^p_{L_p(\Gamma_k)}\right)^{2/p} \lesssim \left(\sum_{j\in\mathcal{N}} |\Gamma_j|\rho_j^{-d}\right)^{2m/d}\|v\|^2_{H^m(\Gamma)},$$

which, again using (3.2) and (5.3), gives the lower bound for $\lambda_{\min}(\boldsymbol{B}')$. □

REMARK 5.3. *The left-hand side of the first inequality in Lemma 5.2 and the right-hand side of the first inequality in Lemma 5.1 should be interpreted as the appropriate limit when $m \to 0$. Observe that these lemmas reproduce several known results as special cases. For example, putting $m = 0$, we obtain estimates for the "mass matrix" corresponding to an operator of order 0:*

$$\min_{k\in\mathcal{N}} |\Gamma_k| \lesssim \lambda_{\min}(\boldsymbol{B}) \le \lambda_{\max}(\boldsymbol{B}) \lesssim \max_{k\in\mathcal{N}} |\Gamma_k| \quad \textit{and} \quad 1 \lesssim \lambda_{\min}(\boldsymbol{B}') \le \lambda_{\max}(\boldsymbol{B}') \lesssim 1 .$$

These are well-known, at least for the shape-regular case (i.e. $|\Gamma_k| \sim \rho_k^d \sim h_k^d$). Moreover in the shape-regular case for general m and d, it is easy to see that Lemmas 5.2 and 5.1, imply the previously proved estimate (1.7)*. Lemmas 5.2 and 5.1 can be combined with* (5.8) *and* (5.7) *to obtain general spectral estimates in the non-shape regular case, in terms of the computable quantities $|\Gamma_k|$ and ρ_k and generic mesh-independent constants. In what follows we shall illustrate the uses of these estimates for operators of general order m, but (since boundary integral equations is our main application), we shall restrict this illustration to the case*

$$d = 2 \quad \textit{and} \quad |\Gamma_k| \sim h_k\rho_k , \quad \textit{for each} \quad k \in \mathcal{N} . \tag{5.18}$$

Then we have the following corollary for operators of general order m:

COROLLARY 5.4. *(i) Assume* (5.18)*. For $-1 < m \le 0$,*

$$\min_{k\in\mathcal{N}}(h_k\rho_k^{1-2m}) \lesssim \lambda_{\min}(\boldsymbol{B}) \le \lambda_{\max}(\boldsymbol{B}) \lesssim \left(\sum_{k\in\mathcal{N}} (h_k\rho_k)^{1-1/m}\right)^{-m} ,$$

$$\min_{k\in\mathcal{N}}(h_k\rho_k^{-1})^m \lesssim \lambda_{\min}(\boldsymbol{B}') \le \lambda_{\max}(\boldsymbol{B}') \lesssim \left(\sum_{k\in\mathcal{N}} h_k\rho_k^{-1}\right)^{-m} .$$

(ii) For $0 \le m < 1$,

$$\left(\sum_{k\in\mathcal{N}} (h_k\rho_k)^{1-1/m}\right)^{-m} \lesssim \lambda_{\min}(\boldsymbol{B}) \le \lambda_{\max}(\boldsymbol{B}) \lesssim \max_{k\in\mathcal{N}} \left(h_k\rho_k^{1-2m}\right) ,$$

$$\left(\sum_{k\in\mathcal{N}} h_k\rho_k^{-1}\right)^{-m} \lesssim \lambda_{\min}(\boldsymbol{B}') \le \lambda_{\max}(\boldsymbol{B}') \lesssim \max_{k\in\mathcal{N}} \left(h_k\rho_k^{-1}\right)^m .$$

The hypersingular and weakly-singular examples considered in §2.1 are then obtained from the special cases $m = 1/2$ and $m = -1/2$. These estimates can be applied to any mesh specified by the user. To illustrate its use on a typical class class of meshes, let us consider the following example.

EXAMPLE 5.5. *Suppose Γ is a plane convex polygon with perimeter γ. For some fixed $\delta > 0$, let $\gamma^{\|}$ be the inscribed polygon, each of whose edges $e^{\|}$ is parallel to and a perpendicular distance δ from a corresponding edge e of γ. To mesh Γ, extend each $e^{\|}$ in a straight line at each end until it touches γ. For δ sufficiently small, this subdivides Γ into near-vertex rhombi, near edge trapezia and an inner polygon. For each e, draw $n-1$ parallel lines inside Γ, a perpendicular distance $(i/n)^\beta$ from e, for $i = 1, \ldots, n-1$. (The last of these lines is an extension of $e^{\|}$.) This defines a mesh of quadrilaterals on each of the rhombi. For each near-edge trapezium, introduce a quadrilateral mesh by subdividing each of its parallel sides uniformly with n subintervals and draw straight lines between corresponding points. Finally subdivide the interior polygon Γ^{int} with a quasiuniform mesh with $O(n^2)$ (triangular or quadrilateral) elements, whose nodes on γ coincide with the nodes already specified. (See Figure 5.1.) This mesh has $N = O(n^2)$ elements, and the mesh in Figure 2.1 is a particular case.*

LEMMA 5.6. *For the class of meshes specified in Example 5.5 we have*

$$\sum_{k\in\mathcal{N}} h_k\rho_k^{-1} \lesssim \begin{cases} N & \textit{for } 1 \le \beta < 2, \\ N(1+\log N) & \textit{for } \beta = 2, \\ N^{\beta/2} & \textit{for } \beta > 2, \end{cases} \quad \textit{and} \quad \sum_{k\in\mathcal{N}} (\rho_k h_k)^{-1} \lesssim \begin{cases} N^2, & \textit{for } 1 \le \beta < 2, \\ N^2(1+\log N)^2, & \beta = 2, \\ N^\beta, & \textit{for } \beta > 2. \end{cases} \tag{5.19}$$

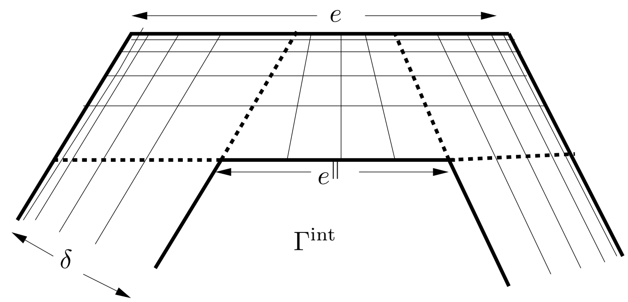

FIG. 5.1. *Section of mesh on polygon* Γ*, depicting two near-vertex rhombi and a near-edge trapezium.*

Proof. For the quasiuniform mesh on $\Gamma^{\rm int}$ the required estimates follow from the standard inequalities $\rho_k \gtrsim h_k \gtrsim N^{-1/2}$. Therefore we only have to consider the near-vertex rhombi and the near-edge trapezia.

Any typical near-vertex rhombus is the image of the unit square $[0,1]^2$ under an invertible affine map. Moreover the mesh on any near-vertex rhombus can be obtained by applying this affine map to the tensor product mesh with vertices $\big((i/n)^\beta, (j/n)^\beta\big)$. Without loss of generality, we can estimate the quantities (5.19) for this unit square because mesh-independent constants are not important. In this case, with $t_j = (j/n)^\beta$ we have $\Delta t_j = t_j - t_{j-1} \simeq n^{-1}(j/n)^{\beta-1}$ and so using the notation from §2.2, in particular (2.7), we have

$$\sum_{k\in\mathcal{N}} h_k\rho_k^{-1} \simeq \sum_{i=1}^n\sum_{j=1}^i \frac{\Delta t_i}{\Delta t_j} \simeq \sum_{i=1}^n\sum_{j=1}^i \frac{(i/n)^{\beta-1}}{(j/n)^{\beta-1}}$$
$$= n^2\sum_{i=1}^n \left(\frac{i}{n}\right)^{\beta-1}\frac{1}{n}\sum_{j=1}^i\left(\frac{j}{n}\right)^{1-\beta}\frac{1}{n} \simeq n^2\int_{1/n}^1 s^{\beta-1}\int_{1/n}^s t^{1-\beta}\,dt\,ds\ ,$$

from which the left-hand inequality in (5.19) follows (on recalling that $N\sim n^2$). Similarly,

$$\sum_{k\in\mathcal{N}} (\rho_k h_k)^{-1} \simeq \sum_{i=1}^n\sum_{j=1}^n \frac{1}{\Delta t_i\,\Delta t_j} \simeq \sum_{i=1}^n\sum_{j=1}^i n^2\left(\frac{i}{n}\right)^{1-\beta}\left(\frac{j}{n}\right)^{1-\beta}$$
$$\simeq n^4\sum_{i=1}^n \left(\frac{i}{n}\right)^{1-\beta}\frac{1}{n}\sum_{j=1}^i\left(\frac{j}{n}\right)^{1-\beta}\frac{1}{n} \simeq n^4\int_{1/n}^1 s^{1-\beta}\int_{1/n}^s t^{1-\beta}\,dt\,ds,$$

from which the right-hand inequality in (5.19) follows.

The meshes on the near-edge trapezia can be obtained as images of the unit square under a non-singular bilinear map, meshed with the tensor product mesh $\big(i/n, (j/n)^\beta\big)$ and the estimates (5.19) are then obtained analogously to above. ∎

The following theorems now follow by combining Corollary 5.4 with Lemma 5.6.

THEOREM 5.7. *Consider the weakly-singular boundary integral equation* (2.1) *on the polygon* Γ *discretised as in Example 5.5. Then for conforming boundary elements of any degree in* $\widetilde{H}^{-1/2}(\Gamma)$ *and with the nodal basis introduced in §3,*

1. *the Galerkin matrix* $\boldsymbol{B}$ *satisfies the spectral bounds* $\lambda_{\max}(\boldsymbol{B}) \lesssim N^{-1}$ *and* $\lambda_{\min}(\boldsymbol{B}) \gtrsim N^{-3\beta/2}$*,*
2. *the diagonally-scaled Galerkin matrix* $\boldsymbol{B}'$ *satisfies*

$$\lambda_{\max}(\boldsymbol{B}') \lesssim \begin{cases} N^{1/2} & \textit{for } 1\le\beta<2,\\ N^{1/2}(1+\log N)^{1/2} & \textit{for } \beta=2,\\ N^{\beta/4} & \textit{for } \beta>2,\end{cases} \qquad \textit{and} \quad \lambda_{\min}(\boldsymbol{B}') \gtrsim N^{-(\beta-1)/4}.$$

Proof. Elementary estimates for the meshes in Example 5.5 yield, for each $k\in\mathcal{N}$,

$$N^{-\beta/2} \lesssim \rho_k \le h_k \lesssim N^{-1/2}\ . \tag{5.20}$$

We apply Corollary 5.4 with $m=-1/2$. The bounds for $\lambda_{\max}(\boldsymbol{B})$ and $\lambda_{\min}(\boldsymbol{B})$ and the lower bound for $\lambda_{\min}(\boldsymbol{B}')$ follow immediately from (5.20), whereas the upper bound for $\lambda_{\max}(\boldsymbol{B}')$ follows from Lemma 5.6. ∎

THEOREM 5.8. *Consider the hypersingular boundary integral equation* (2.3) *on the polygonal screen* Γ *discretised as in Example 5.5. For conforming boundary elements of any degree in* $\widetilde{H}^{1/2}(\Gamma)$ *and with the nodal basis introduced in §3,*

1. *the Galerkin matrix* $\boldsymbol{B}$ *satisfies the spectral bounds* $\lambda_{\max}(\boldsymbol{B}) \lesssim N^{-1/2}$ *and*

$$\lambda_{\min}(\boldsymbol{B}) \gtrsim \begin{cases} N^{-1}, & \text{for } 1 \le \beta < 2, \\ N^{-1}(1+\log N)^{-1}, & \text{for } \beta = 2, \\ N^{-\beta/2}, & \text{for } \beta > 2, \end{cases}$$

2. *the diagonally-scaled Galerkin matrix* $\boldsymbol{B}'$ *satisfies* $\lambda_{\max}(\boldsymbol{B}') \lesssim N^{(\beta-1)/4}$ *and*

$$\lambda_{\min}(\boldsymbol{B}') \gtrsim \begin{cases} N^{-1/2}, & \text{for } 1 \le \beta < 2, \\ N^{-1/2}(1+\log N)^{-1/2}, & \text{for } \beta = 2, \\ N^{-\beta/4}, & \text{for } \beta > 2. \end{cases}$$

Proof. Note that the condition that the finite element space is $\widetilde{H}^{1/2}(\Gamma)-$conforming implies that it must be chosen from the class $\{v \in \mathcal{S}_1^\ell(\mathcal{P}) : v|_{\partial\Gamma} = 0\}$, for some $\ell \ge 1$. We apply Corollary 5.4 with $m = 1/2$. The upper bounds for $\lambda_{\max}(\boldsymbol{B})$ and $\lambda_{\max}(\boldsymbol{B}')$ follow immediately from (5.20), whereas the lower bounds follow from Lemma 5.6. ∎

6. Numerical Experiments. In this section we report some numerical experiments with the integral equations from §2 on the square screen (2.5), with the power graded meshes (2.8).

First we consider the weakly-singular equation discretised using piecewise-constant basis functions. For $\beta = 2$ and $\beta = 3$, Tables 6.1 and 6.2 show the extremal eigenvalues and the condition numbers of $\boldsymbol{B}$ and of $\boldsymbol{B}'$. From one row of the table to the next, the number of subintervals along each axis doubles so the number of degrees of freedom N increases by a factor of 4. For each of the six quantities under investigation, the left-hand column shows the quantity itself whereas the right-hand column gives the apparent exponent μ such that the quantity is proportional to N^μ. (To compute μ, we simply divide the logarithm of the ratio of successive values by $\log 4$.) The observed exponent values indicate that the estimates of Theorem 5.7 are sharp for $\boldsymbol{B}$ but not for $\boldsymbol{B}'$. However, the improved spectral bounds for $\boldsymbol{B}'$, proved later in Theorem 7.4, appear to be sharp up to logarithmic factors. We remark that $\beta = 3$ gives the optimal convergence rate $O(N^{-3})$ for the capacitance of Γ, when piecewise constant elements are used (see, e.g. [6]).

Our second experiment is for the hypersingular equation discretised using continuous piecewise-bilinear basis functions. Tables 6.3 and 6.4 give our results for $\beta = 2$ and $\beta = 3$, which indicate that the estimates in Theorem 5.8 are (essentially) sharp for $\boldsymbol{B}$ but not for $\boldsymbol{B}'$. However, the improved spectral bounds for $\boldsymbol{B}'$, proved later in Theorem 7.5, appear to be sharp up to logarithmic factors.

The remainder of the paper is devoted to explaining our numerical results for $\boldsymbol{B}'$.

7. Sharper Results for Special Cases.

7.1. Improved Spectral Bounds for $\boldsymbol{B}'$. For each of the model problems of Section 2, the observed rate of growth for $\text{cond}(\boldsymbol{B}')$ is slower than the rate predicted by the results proved in Section 5 if the mesh grading is sufficiently strong, more precisely, if $\beta > 2$. However, the next lemma leads to bounds that are sharp to within logarithmic factors for $\lambda_{\max}(\boldsymbol{B}')$ (weakly-singular case) and for $\lambda_{\min}(\boldsymbol{B}')$ (hypersingular case). Recall that γ denotes the perimeter of the open surface Γ.

LEMMA 7.1. *Let* $d_k = \sup_{x\in\Gamma_k} \text{dist}(x, \gamma)$ *and assume that* $d_{\min} := \min_{k\in\mathcal{N}} d_k$ *is sufficiently small.*

1. *For the weakly-singular boundary integral equation* (2.1) *on an open surface discretised with conforming finite elements of any degree in* $\widetilde{H}^{-1/2}(\Gamma)$,

$$\lambda_{\max}(\boldsymbol{B}') \lesssim \left(\log \frac{1}{d_{\min}}\right)^2 \max_{j\in\mathcal{N}} \frac{d_j}{\rho_j}.$$

2. *For the hypersingular boundary integral equation* (2.3) *on an open surface discretised with conforming finite elements of any degree in* $\widetilde{H}^{1/2}(\Gamma)$,

$$\lambda_{\min}(\boldsymbol{B}') \gtrsim \left(\log \frac{1}{d_{\min}}\right)^{-2} \min_{j\in\mathcal{N}} \frac{\rho_j}{d_j}.$$

N	$\lambda_{\max}(\boldsymbol{B})$		$\lambda_{\min}(\boldsymbol{B})$		$\mathrm{cond}(\boldsymbol{B})$	
64	9.89E-02	-0.896	7.54E-05	-2.991	1.31E+03	2.095
256	2.57E-02	-0.973	1.18E-06	-3.000	2.18E+04	2.026
1024	6.48E-03	-0.993	1.84E-08	-3.000	3.52E+05	2.007
4096	1.62E-03	-0.998	2.88E-10	-3.000	5.64E+06	2.002
16384	4.06E-04	-1.000	4.50E-12	-3.000	9.03E+07	2.000
Theorem 5.7	$\lesssim N^{-1}$		$\gtrsim N^{-3}$		$\lesssim N^2$	

N	$\lambda_{\max}(\boldsymbol{B}')$		$\lambda_{\min}(\boldsymbol{B}')$		$\mathrm{cond}(\boldsymbol{B}')$	
64	7.09E+00	0.477	3.53E-01	-0.060	2.01E+01	0.537
256	1.40E+01	0.492	3.20E-01	-0.071	4.38E+01	0.563
1024	2.79E+01	0.497	2.71E-01	-0.121	1.03E+02	0.618
4096	5.58E+01	0.499	2.31E-01	-0.115	2.42E+02	0.615
16384	1.12E+02	0.500	1.99E-01	-0.105	5.59E+02	0.605
Theorem 5.7	$\lesssim N^{1/2}(1+\log N)^{1/2}$		$\gtrsim N^{-1/4}$		$\lesssim N^{3/4}(1+\log N)^{1/2}$	
Theorem 7.4	$\lesssim N^{1/2}(1+\log N)^{1/2}$		$\gtrsim (1+\log N)^{-1}$		$\lesssim N^{1/2}(1+\log N)^{3/2}$	

TABLE 6.1
Weakly singular integral equation (2.1) *on the screen* (2.5) *with* $\beta = 2$.

N	$\lambda_{\max}(\boldsymbol{B})$		$\lambda_{\min}(\boldsymbol{B})$		$\mathrm{cond}(\boldsymbol{B})$	
64	1.78E-01	-0.755	1.28E-06	-4.496	1.39E+05	3.741
256	4.90E-02	-0.932	2.50E-09	-4.500	1.96E+07	3.568
1024	1.26E-02	-0.982	4.89E-12	-4.500	2.57E+09	3.518
4096	3.16E-03	-0.995	9.54E-15	-4.500	3.31E+11	3.505
16384	7.91E-04	-0.999	1.86E-17	-4.500	4.25E+13	3.502
Theorem 5.7	$\lesssim N^{-1}$		$\gtrsim N^{-9/2}$		$\lesssim N^{7/2}$	

N	$\lambda_{\max}(\boldsymbol{B}')$		$\lambda_{\min}(\boldsymbol{B}')$		$\mathrm{cond}(\boldsymbol{B}')$	
64	6.43E+00	0.463	3.25E-01	-0.096	1.98E+01	0.559
256	1.26E+01	0.488	2.45E-01	-0.204	5.16E+01	0.692
1024	2.51E+01	0.496	1.86E-01	-0.200	1.35E+02	0.696
4096	5.02E+01	0.499	1.48E-01	-0.165	3.39E+02	0.663
16384	1.00E+02	0.499	1.22E-01	-0.137	8.21E+02	0.637
Theorem 5.7	$\lesssim N^{3/4}$		$\gtrsim N^{-1/2}$		$\lesssim N^{5/4}$	
Theorem 7.4	$\lesssim N^{1/2}(1+\log N)^2$		$\gtrsim (1+\log N)^{-1}$		$\lesssim N^{1/2}(1+\log N)^3$	

TABLE 6.2
Weakly singular integral equation (2.1) *on the screen* (2.5) *with* $\beta = 3$.

Proof. First we prove part 2. Let $v \in X \subset \widetilde{H}^{1/2}(\Gamma)$ and decompose v as in (5.1). Taking $p = 2$ in (5.17) we have

$$\|v_k\|^2_{\widetilde{H}^m(\Gamma)} \le \rho_k^{-2m} \|v\|^2_{L_2(\Gamma_k)} \quad \text{for } 0 \le m \le 1 \text{ and } 0 \le 2m < d. \tag{7.1}$$

Now define $w(x)$ by $w(x) = \mathrm{dist}(x, \gamma)$. It can be shown [9, Lemma 3.32] that

$$\|vw^{-s}\|^2_{L_2(\Gamma)} = \int_\Gamma w(x)^{-2s} v(x)^2\, dx \lesssim \frac{1}{(\frac{1}{2} - s)^2} \|v\|^2_{H^s(\Gamma)} \quad \text{for } \tfrac{1}{4} \le s < \tfrac{1}{2}, \tag{7.2}$$

where the hidden constant is independent of $s \in [\frac{1}{4}, \frac{1}{2})$. Since $w(x) \lesssim d_k$ for $x \in \Gamma_k$, using (7.1) with $m = 1/2$, we obtain

$$\|v_k\|^2_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \rho_k^{-1} \|v\|^2_{L_2(\Gamma_k)} \lesssim \rho_k^{-1} \|(d_k/w)^s v\|^2_{L_2(\Gamma_k)} = \frac{d_k^{2s}}{\rho_k} \|vw^{-s}\|^2_{L_2(\Gamma_k)}\ , \quad s > 0.$$

N	$\lambda_{\max}(\boldsymbol{B})$		$\lambda_{\min}(\boldsymbol{B})$		$\text{cond}(\boldsymbol{B})$	
49	1.00E-01	-0.342	2.27E-02	-0.798	4.41E+00	0.456
225	5.78E-02	-0.397	5.69E-03	-1.000	1.02E+01	0.602
961	3.14E-02	-0.440	1.42E-03	-1.000	2.21E+01	0.560
3969	1.65E-02	-0.465	3.56E-04	-1.000	4.63E+01	0.535
16129	8.49E-03	-0.479	8.89E-05	-1.000	9.54E+01	0.521
Theorem 5.8	$\lesssim N^{-1/2}$		$\gtrsim N^{-1}(1+\log N)^{-1}$		$\lesssim N^{1/2}(1+\log N)$	
N	$\lambda_{\max}(\boldsymbol{B}')$		$\lambda_{\min}(\boldsymbol{B}')$		$\text{cond}(\boldsymbol{B}')$	
49	1.64E+00	0.156	5.97E-01	0.055	2.74E+00	0.101
225	1.84E+00	0.084	4.28E-01	-0.240	4.29E+00	0.323
961	1.94E+00	0.041	2.18E-01	-0.487	8.92E+00	0.528
3969	1.99E+00	0.018	1.09E-01	-0.497	1.82E+01	0.515
16129	2.02E+00	0.008	5.48E-02	-0.499	3.68E+01	0.508
Theorem 5.8	$\lesssim N^{1/4}$		$\gtrsim N^{-1/2}(1+\log N)^{-1/2}$		$\lesssim N^{3/4}(1+\log N)^{1/2}$	
Theorem 7.5	$\lesssim 1$		$\gtrsim N^{-1/2}(1+\log N)^{-1/2}$		$\lesssim N^{1/2}(1+\log N)^{1/2}$	

TABLE 6.3
Hypersingular integral equation (2.3) *on the screen* (2.5) *with* $\beta = 2$.

N	$\lambda_{\max}(\boldsymbol{B})$		$\lambda_{\min}(\boldsymbol{B})$		$\text{cond}(\boldsymbol{B})$	
49	1.38E-01	-0.348	1.52E-02	-1.189	9.12E+00	0.840
225	8.47E-02	-0.354	1.90E-03	-1.500	4.47E+01	1.146
961	4.72E-02	-0.422	2.37E-04	-1.500	1.99E+02	1.078
3969	2.51E-02	-0.456	2.96E-05	-1.500	8.46E+02	1.044
16129	1.30E-02	-0.473	3.70E-06	-1.500	3.51E+03	1.027
Theorem 5.8	$\lesssim N^{-1/2}$		$\gtrsim N^{-3/2}$		$\lesssim N$	
N	$\lambda_{\max}(\boldsymbol{B}')$		$\lambda_{\min}(\boldsymbol{B}')$		$\text{cond}(\boldsymbol{B}')$	
49	1.68E+00	0.104	5.40E-01	0.043	3.11E+00	0.061
225	1.87E+00	0.078	4.30E-01	-0.165	4.36E+00	0.243
961	1.96E+00	0.033	2.72E-01	-0.330	7.21E+00	0.363
3969	2.00E+00	0.015	1.37E-01	-0.494	1.46E+01	0.509
16129	2.02E+00	0.006	6.87E-02	-0.498	2.94E+01	0.505
Theorem 5.8	$\lesssim N^{1/2}$		$\gtrsim N^{-1/2}(1+\log N)^{-2}$		$\lesssim N(1+\log N)^2$	
Theorem 7.5	$\lesssim 1$		$\gtrsim N^{-1/2}(1+\log N)^{-2}$		$\lesssim N^{1/2}(1+\log N)^2$	

TABLE 6.4
Hypersingular integral equation (2.3) *on the screen* (2.5) *with* $\beta = 3$.

Hence, using (7.2), we have

$$\sum_{k\in\mathcal{N}} \|v_k\|^2_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \left(\max_{j\in\mathcal{N}} \frac{d_j^{2s}}{\rho_j}\right) \sum_{k\in\mathcal{N}} \|vw^{-s}\|^2_{L_2(\Gamma_k)}$$

$$\lesssim \left(\max_{j\in\mathcal{N}} \frac{d_j^{2s}}{\rho_j}\right) \|vw^{-s}\|^2_{L_2(\Gamma)} \lesssim \frac{1}{(\frac{1}{2}-s)^2} \left(\max_{j\in\mathcal{N}} \frac{d_j^{2s}}{\rho_j}\right) \|v\|^2_{H^s(\Gamma)} \ ,$$

where hidden the constants are independent of $s \in [\frac{1}{4}, \frac{1}{2})$. Now with $\epsilon := (\log 1/d_{\min})^{-1}$, it follows that $d_j^{-2\epsilon} \lesssim 1$ for all $j \in \mathcal{N}$. Hence putting $s = \frac{1}{2} - \epsilon$, we obtain

$$\sum_{k\in\mathcal{N}} \|v_k\|^2_{\widetilde{H}^{1/2}(\Gamma)} \lesssim \left(\log \frac{1}{d_{\min}}\right)^2 \left(\max_{j\in\mathcal{N}} \frac{d_j}{\rho_j}\right) \|v\|^2_{H^{1/2}(\Gamma)}.$$

The estimate in part 2 follows at once.

To prove part 1, we apply a duality argument. Suppose $\frac{1}{4} \le s < \frac{1}{2}$. Then applying Cauchy–Schwarz together with (7.2) we obtain

$$\left|\langle v,\psi\rangle_{L_2(\Gamma)}\right| = \left|\langle vw^s,\psi w^{-s}\rangle_{L_2(\Gamma)}\right| \lesssim \|vw^s\|_{L_2(\Gamma)}\|\psi w^{-s}\|_{L_2(\Gamma)} \lesssim \frac{1}{\frac{1}{2}-s}\|vw^s\|_{L_2(\Gamma)}\|\psi\|_{H^s(\Gamma)}\ ,$$

and hence $\|v\|_{\widetilde{H}^{-s}(\Gamma)} \lesssim (\frac{1}{2}-s)^{-1}\|vw^s\|_{L_2(\Gamma)}$. Recalling (5.1) and (5.4), this implies that

$$\|v\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \|v\|^2_{\widetilde{H}^{-s}(\Gamma)} \lesssim (\tfrac{1}{2}-s)^{-2}\|vw^s\|^2_{L_2(\Gamma)} \lesssim (\tfrac{1}{2}-s)^{-2}\sum_{k\in\mathcal{N}}\|v_kw^s\|^2_{L_2(\Gamma)}. \tag{7.3}$$

Taking $p=2$ in Theorem 4.2 (i), and then using Theorem 4.2 (iii) and (3.2), we see that

$$\begin{aligned}\|v_kw^s\|^2_{L_2(\Gamma)} &\lesssim d_k^{2s}\alpha_k^2\|\phi_k\|^2_{L_2(\Gamma)} \simeq \frac{d_k^{2s}}{\rho_k}\,\alpha_k^2|\Gamma_k|\rho_k\\ &\lesssim \frac{d_k^{2s}}{\rho_k}\,\alpha_k^2\|\phi_k\|^2_{\widetilde{H}^{-1/2}(\Gamma)} = \frac{d_k^{2s}}{\rho_k}\,\|v_k\|^2_{\widetilde{H}^{-1/2}(\Gamma)}, \quad \text{where } \alpha_k = v(x_k)\ .\end{aligned} \tag{7.4}$$

So, combining (7.3) and (7.4) and putting $s=\frac{1}{2}-\epsilon$ with $\epsilon = (\log 1/d_{\min})^{-1}$, as above, we obtain

$$\|v\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim \left(\log\frac{1}{d_{\min}}\right)^2\left(\max_{j\in\mathcal{N}}\frac{d_j}{\rho_j}\right)\sum_{k\in\mathcal{N}}\|v_k\|^2_{\widetilde{H}^{-1/2}(\Gamma)},$$

which proves part 1.
∎

7.2. Improved bounds for $\Phi_{m,k}$. For the rest of the paper, we restrict our attention to piecewise-constant and continuous piecewise-bilinear basis functions on power-graded tensor-product meshes as defined in §2.2.

We saw in (5.7) that $\Phi_{m,k} \lesssim 1$ if $-1 \le m \le 0$. The next lemma gives a sharp two-sided bound for the special case that occurs in our numerical experiments.

LEMMA 7.2. *For the piecewise-constant nodal basis on a rectangular mesh,*

$$1 \gtrsim \Phi_{-1/2,k} \gtrsim \frac{1}{1+\log(h_k/\rho_k)}.$$

Proof. We may assume without loss of generality that $\Gamma_k = [-h_k/2, h_k/2]\times[-\rho_k/2,\rho_k/2]$. For brevity we omit the subscript k for the remainder of the proof. Define the 1D piecewise-constant basis function

$$\psi(x,h) = \begin{cases}1 & \text{for } -h/2 < x < h/2,\\ 0 & \text{otherwise,}\end{cases}$$

and write the tensor-product basis function as $\phi(x) = \psi(x_1,h)\psi(x_2,\rho)$. Recalling (5.7), we see that the result will follow from the upper bound

$$\|\phi\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \lesssim h\rho^2\left(1+\log\frac{h}{\rho}\right). \tag{7.5}$$

Denote the 2D Fourier transform of ϕ by

$$\hat{\phi}(\xi) = \int_{\mathbb{R}^2} e^{-i2\pi\xi\cdot x}\phi(x)\,dx = \hat{\psi}(\xi_1,h)\hat{\psi}(\xi_2,\rho),$$

where $\hat{\psi}$, the 1D Fourier transform of ψ, is given by

$$\hat{\psi}(\xi_1,h) = \int_{-h/2}^{h/2} e^{-i2\pi\xi_1x_1}\,dx_1 = h\,\mathrm{sinc}(\xi_1h), \qquad \mathrm{sinc}(z) = \begin{cases}\dfrac{\sin\pi z}{\pi z}, & z\ne 0,\\ 1, & z=0.\end{cases}$$

Note that $|\hat\psi(\xi,h)| \le \min(h,|\xi|^{-1})$. We have the norm equivalence

$$\|\phi\|^2_{\widetilde H^{-1/2}(\Gamma)} = \|\phi\|^2_{\widetilde H^{-1/2}(\mathbb{R}^2)} \simeq \int_{\mathbb{R}^2}(1+|\xi|^2)^{-1/2}|\hat\phi(\xi)|^2\,d\xi$$
$$= \int_{-\infty}^{\infty}|\hat\psi(\xi_1,h)|^2\int_{-\infty}^{\infty}(1+\xi_1^2+\xi_2^2)^{-1/2}|\hat\psi(\xi_2,\rho)|^2\,d\xi_2\,d\xi_1 =: I_1+I_2+I_3+I_4+I_5,$$

with

$$I_1 = \int_{\substack{-\infty<\xi_1<\infty\\ |\xi_2|>\rho^{-1}}}, \quad I_2 = \int_{\substack{|\xi_1|<h^{-1}\\ |\xi_2|<h^{-1}}}, \quad I_3 = \int_{\substack{|\xi_1|<h^{-1}\\ h^{-1}<|\xi_2|<\rho^{-1}}}, \quad I_4 = \int_{\substack{|\xi_1|>h^{-1}\\ |\xi_2|<h^{-1}}}, \quad I_5 = \int_{\substack{|\xi_1|>h^{-1}\\ h^{-1}<|\xi_2|<\rho^{-1}}}.$$

By Plancherel's theorem,

$$I_1 \le \int_{-\infty}^{\infty}|\hat\psi(\xi_1,h)|^2\,d\xi_1\int_{|\xi_2|>\rho^{-1}}|\xi_2|^{-1}|\hat\psi(\xi_2,\rho)|^2\,d\xi_2 \le 2\int_{-\infty}^{\infty}|\psi(x_1,h)|^2\,dx_1\int_{\rho^{-1}}^{\infty}\xi_2^{-3}\,d\xi_2 = h\rho^2.$$

Using polar coordinates we find that

$$I_2 \le \int_{-h^{-1}}^{h^{-1}}h^2\int_{-h^{-1}}^{h^{-1}}(1+\xi_1^2+\xi_2^2)^{-1/2}\rho^2\,d\xi_2\,d\xi_1 \le h^2\rho^2\int_0^{h^{-1}\sqrt2}(1+r^2)^{-1/2}2\pi r\,dr \le 2\pi\sqrt2\,h\rho^2,$$

and simple estimation gives

$$I_3 \le 4\int_0^{h^{-1}}h^2\,d\xi_1\int_{h^{-1}}^{\rho^{-1}}\xi_2^{-1}\rho^2\,d\xi_2 = 4h\rho^2\int_{h^{-1}}^{\rho^{-1}}\frac{d\xi_2}{\xi_2} = 4h\rho^2\log\frac{h}{\rho},$$
$$I_4 \le 4\int_{h^{-1}}^{\infty}\xi_1^{-2}\int_0^{h^{-1}}\xi_1^{-1}\rho^2\,d\xi_2\,d\xi_1 = 4h^{-1}\rho^2\int_{h^{-1}}^{\infty}\xi_1^{-3}\,d\xi_1 = 2h\rho^2,$$
$$I_5 \le 4\int_{h^{-1}}^{\infty}\xi_1^{-2}\,d\xi_1\int_{h^{-1}}^{\rho^{-1}}\xi_2^{-1}\rho^2\,d\xi_2 = 4h\rho^2\log\frac{h}{\rho}.$$

□

LEMMA 7.3. *Let Γ be the square screen (2.5). For the continuous, piecewise-bilinear nodal basis on the power-graded mesh with vertices defined by (2.8), we have*

$$\Phi_{m,k} \simeq 1 \quad \textit{for } 0 \le m \le 1.$$

Proof. Without loss of generality, we may assume that x_k is the origin and that $\Gamma_k = [-h_-,h_+]\times[-\rho_-,\rho_+]$ with $h_\pm \simeq h_k$ and $\rho_\pm \simeq \rho_k$. We define the 1D continuous, piecewise-linear basis function on the interval $(-h_-,h_+)$,

$$\psi(x,h_+,h_-) = \begin{cases} 1+\dfrac{x}{h_-} & \text{for } -h_- < x < 0,\\ 1-\dfrac{x}{h_+} & \text{for } 0 < x < h_+,\\ 0 & \text{otherwise,}\end{cases}$$

and write the bilinear basis function on Γ_k by $\phi(x) = \psi(x_1,h_+,h_-)\psi(x_2,\rho_+,\rho_-)$. Recalling (5.8), we see that the result will follow from the lower bound

$$\|\phi\|^2_{\widetilde H^m(\Gamma)} \gtrsim h\rho^{1-2m}. \tag{7.6}$$

Since $\|\phi\|^2_{L_2(\Gamma)} \simeq h\rho$ and $|\phi|^2_{H^1(\Gamma)} \simeq h\rho(h^{-2}+\rho^{-2})$, the cases $m=0$ and $m=1$ are obvious. If $0<m<1$ then we use the norm equivalence

$$\|\phi\|^2_{\widetilde H^m(\Gamma)} = \|\phi\|^2_{\widetilde H^m(\mathbb{R}^2)} \simeq \int_{\mathbb{R}^2}(1+|\xi|^2)^m|\hat\phi(\xi)|^2\,d\xi,$$

where

$$\hat{\phi}(\xi) = \hat{\psi}(\xi_1, h_+, h_-)\hat{\psi}(\xi_2, \rho_+, \rho_-) \quad \text{and} \quad \hat{\psi}(\xi, h_+, h_-) = \frac{1}{(2\pi\xi)^2}\left\{\frac{1 - e^{i2\pi\xi h_-}}{h_-} + \frac{1 - e^{-i2\pi\xi h_+}}{h_+}\right\}.$$

Since $(1 + |\xi|^2)^m \geq |\xi_2|^{2m}$ we see that

$$\|\phi\|^2_{\widetilde{H}^m(\mathbb{R}^2)} \gtrsim \int_{\mathbb{R}^2} |\xi_2|^{2m} |\hat{\phi}(\xi)|^2 \, d\xi = \int_{-\infty}^{\infty} \left|\hat{\psi}(\xi_1, h_+, h_-)\right|^2 d\xi_1 \int_{-\infty}^{\infty} |\xi_2|^{2m} \left|\hat{\psi}(\xi_2, \rho_+, \rho_-)\right|^2 d\xi_2, \tag{7.7}$$

and by Plancherel's theorem,

$$\int_{-\infty}^{\infty} \left|\hat{\psi}(\xi_1, h_+, h_-)\right|^2 d\xi_1 = \int_{-\infty}^{\infty} \left|\psi(x_1, h_+, h_-)\right|^2 dx_1 = \frac{h_+ + h_-}{3}. \tag{7.8}$$

Now define $h = (h_+ + h_-)/2$, $\Delta h = (h_+ - h_-)/2$, $\rho = (\rho_+ + \rho_-)/2$ and $\Delta\rho = (\rho_+ - \rho_-)/2$, so that $h_\pm = h \pm \Delta h$ and $\rho_\pm = \rho \pm \Delta\rho$. Using the substitution $\xi_2 = t/\rho$ in (7.7), we see that

$$\|\phi\|^2_{\widetilde{H}^m(\mathbb{R}^2)} \gtrsim h \int_{-\infty}^{\infty} |t/\rho|^{2m} \left|\hat{\psi}(t/\rho, \rho_+, \rho_-)\right|^2 \frac{dt}{\rho} = h\rho^{1-2m} \int_{-\infty}^{\infty} |t|^{2m} \left|\rho^{-1}\hat{\psi}(t/\rho, \rho_+, \rho_-)\right|^2 dt.$$

Putting $\epsilon = \Delta\rho/\rho = (\rho_+ - \rho_-)/(\rho_+ + \rho_-) \in (-1, 1)$, a simple calculation gives

$$\rho^{-1}\hat{\psi}(t/\rho, \rho_+, \rho_-) = (1 - \epsilon)f_+[(1 - \epsilon)t] + (1 + \epsilon)f_-[(1 + \epsilon)t] \quad \text{where} \quad f_\pm(t) = \frac{1 - e^{\pm i2\pi t}}{(2\pi t)^2}.$$

Since $f_\pm(t) = \mp i/(2\pi t) + O(1)$ as $t \to 0$ and $f_\pm(t) = O(t^{-2})$ as $t \to \infty$, the integral

$$I(\epsilon) = \int_{-\infty}^{\infty} |t|^{2m} \left|\rho^{-1}\hat{\psi}(t/\rho, \rho_+, \rho_-)\right|^2 dt$$

is analytic for $|\epsilon| < 1$, and (since $\rho_- = \rho_+ = \rho$ when $\epsilon = 0$) we have

$$I(0) = \int_{-\infty}^{\infty} |t|^{2m} \left(\frac{\sin \pi t}{\pi t}\right)^4 dt \simeq 1 \quad \text{for } 0 \leq m \leq 1.$$

The lower bound (7.6) follows for $0 < m < 1$ because $\max_{k \in \mathcal{N}} \epsilon_k \to 0$ as $N \to \infty$. □

7.3. Sharper versions of the Theorems in §5.

THEOREM 7.4. *Consider the weakly-singular boundary integral equation* (2.1) *on the square screen* (2.5). *Then for piecewise-constant nodal basis functions with the mesh* (2.8),

$$\lambda_{\max}(\boldsymbol{B}') \lesssim N^{1/2} \times \begin{cases} 1 & \text{if } 1 \leq \beta < 2, \\ (1 + \log N)^{1/2} & \text{if } \beta = 2, \\ (1 + \log N)^2 & \text{if } \beta > 2, \end{cases}$$

and

$$\lambda_{\min}(\boldsymbol{B}') \gtrsim \begin{cases} 1 & \text{if } \beta = 1, \\ (1 + \log N)^{-1} & \text{if } \beta > 1. \end{cases}$$

Proof. Note that $d_{\min}$ (defined in Lemma 7.1) satisfies $d_{\min} = n^{-\beta} \simeq N^{-\beta/2}$. Note also that

$$\max_{k \in \mathcal{N}} \frac{d_k}{\rho_k} = \max_{1 \leq j \leq n/2} \frac{t_j}{\Delta t_j} \simeq \max_{1 \leq j \leq n/2} \frac{(j/n)^\beta}{n^{-1}(j/n)^{\beta - 1}} = \max_{1 \leq j \leq n/2} j \simeq n \simeq N^{1/2}. \tag{7.9}$$

Hence, from part 1 of Lemma 7.1 we obtain

$$\lambda_{\max}(\boldsymbol{B}') \lesssim N^{1/2}(1 + \log N)^2 \quad \text{for all } \beta \geq 1.$$

We can combine this with Theorem 5.7 to obtain the required bounds on $\lambda_{\max}(\boldsymbol{B}')$. The proof is completed by using Lemmas 5.1 and 7.2 to obtain

$$\lambda_{\min}(\boldsymbol{B}') \gtrsim \min_{k\in\mathcal{N}} \Phi_{-1/2,k} \simeq \frac{1}{1+\log(n^{-1}/n^{-\beta})} \simeq \frac{1}{1+(\beta-1)\log n} \simeq \begin{cases} 1, & \text{for } \beta = 1, \\ (1+\log N)^{-1}, & \text{for } \beta > 1. \end{cases}$$

□

THEOREM 7.5. *Consider the hypersingular boundary integral equation* (2.3) *on the square screen* (2.5). *Then with conforming piecewise bilinear nodal basis functions on the mesh* (2.8),

$$\lambda_{\max}(\boldsymbol{B}') \lesssim 1 \quad \textit{and} \quad \lambda_{\min}(\boldsymbol{B}') \gtrsim N^{-1/2} \times \begin{cases} 1, & \textit{for } 1 \le \beta < 2, \\ (1+\log N)^{-1/2}, & \textit{for } \beta = 2, \\ (1+\log N)^{-2}, & \textit{for } \beta > 2. \end{cases}$$

Proof. Lemmas 5.1 and 7.3 imply that $\lambda_{\max}(\boldsymbol{B}') \lesssim \max_{k\in\mathcal{N}} \Phi_{1/2,j} \lesssim 1$. For $\beta > 2$ we sharpen the bound in Theorem 5.8 by using part 2 of Lemma 7.1. In fact, $d_{\min} \simeq n^{-\beta} \simeq N^{-\beta/2}$ and, by (7.9), $\min_{k\in\mathcal{N}} \rho_k/d_k \simeq N^{-1/2}$. Hence $\lambda_{\min}(\boldsymbol{B}') \gtrsim N^{-1/2}/(1+\log N)^2$, completing the proof. □

8. Numerical experiments with a different family of meshes. To conclude, we present some numerical results for the weakly-singular equation (2.1) over the non-convex, polygonal screen

$$\Gamma = (-1,1)^2 \setminus ([0,1] \times [-1,0]). \tag{8.1}$$

The meshes are constructed by a pseudo-adaptive procedure that starts with a uniform mesh and then selectively bisects elements so that the relation between h_K, ρ_K and the distance to the nearest edge or corner is equivalent to that for a power-graded mesh with a chosen grading exponent $\beta \ge 1$. For simplicity, we grade only into the edges and vertex of the re-entrant corner, i.e., we ignore the other four edges and five corners. Figure 8.1 shows a typical mesh. Strictly speaking, our theory does not cover this example because we require conforming meshes, although in practical terms there is no need to enforce any inter-element continuity condition at the hanging nodes because we use discontinuous (piecewise-constant) nodal basis functions. Tables 8.1 and 8.2 give our numerical results using meshes with $\beta = 2$ and $\beta = 3$. The asymptotic behaviour of the extremal eigenvalues and the condition numbers is essentially the same as we observed previously, in Tables 6.1 and 6.2, for simple tensor-product, power-graded meshes on a square screen with the same choices of β.

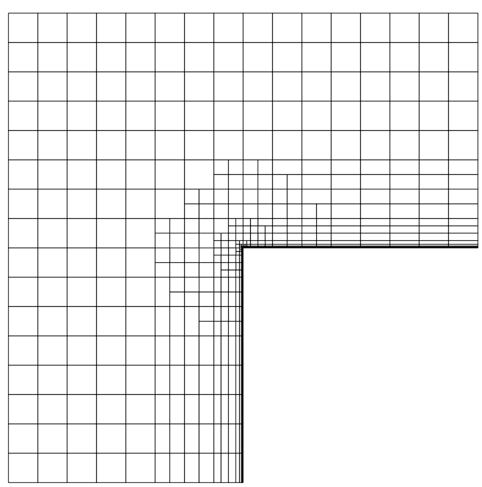

FIG. 8.1. *Anisotropic mesh on non-convex screen, produced by a pseudo-adaptive procedure, $N = 488$, $\beta = 3$*

N	$\lambda_{\max}(\boldsymbol{B})$		$\lambda_{\min}(\boldsymbol{B})$		$\mathrm{cond}(\boldsymbol{B})$	
73	2.82E-01		9.15E-04		3.08E+02	
296	7.15E-02	-0.981	1.43E-05	-2.971	5.00E+03	1.990
1185	1.79E-02	-0.999	2.23E-07	-2.998	8.01E+04	1.999
4743	4.47E-03	-0.999	3.49E-09	-2.999	1.28E+06	1.999
18958	1.12E-03	-1.000	5.46E-11	-3.002	2.05E+07	2.001
Expected	$\lesssim N^{-1}$		$\gtrsim N^{-3}$		$\lesssim N^{2}$	

N	$\lambda_{\max}(\boldsymbol{B}')$		$\lambda_{\min}(\boldsymbol{B}')$		$\mathrm{cond}(\boldsymbol{B}')$	
73	8.40E+00		3.61E-01		2.33E+01	
296	1.67E+01	0.491	3.11E-01	-0.106	5.37E+01	0.597
1185	3.33E+01	0.497	2.67E-01	-0.111	1.25E+02	0.608
4743	6.65E+01	0.499	2.30E-01	-0.108	2.89E+02	0.607
18958	1.33E+02	0.500	2.00E-01	-0.101	6.65E+02	0.601
Expected	$\lesssim N^{1/2}(1+\log N)^{1/2}$		$\gtrsim (1+\log N)^{-1}$		$\lesssim N^{1/2}(1+\log N)^{3/2}$	

TABLE 8.1
Weakly singular integral equation (2.1) *on the non-convex screen* (8.1) *with* $\beta = 2$.

N	$\lambda_{\max}(\boldsymbol{B})$		$\lambda_{\min}(\boldsymbol{B})$		$\mathrm{cond}(\boldsymbol{B})$	
111	2.81E-01		1.44E-05		1.95E+04	
488	6.66E-02	-0.972	2.82E-08	-4.213	2.37E+06	3.241
2017	1.67E-02	-0.976	5.50E-11	-4.396	3.03E+08	3.421
8095	4.21E-03	-0.991	1.07E-13	-4.489	3.92E+10	3.499
Expected	$\lesssim N^{-1}$		$\gtrsim N^{-9/2}$		$\lesssim N^{7/2}$	

N	$\lambda_{\max}(\boldsymbol{B}')$		$\lambda_{\min}(\boldsymbol{B}')$		$\mathrm{cond}(\boldsymbol{B}')$	
111	1.03E+01		2.72E-01		3.78E+01	
488	2.12E+01	0.489	2.04E-01	-0.194	1.04E+02	0.683
2017	4.29E+01	0.497	1.60E-01	-0.169	2.67E+02	0.666
8095	8.75E+01	0.498	1.31E-01	-0.145	6.53E+02	0.643
Expected	$\lesssim N^{1/2}(1+\log N)^{2}$		$\gtrsim (1+\log N)^{-1}$		$\lesssim N^{1/2}(1+\log N)^{3}$	

TABLE 8.2
Weakly singular integral equation (2.1) *on the non-convex screen* (8.1) *with* $\beta = 3$.

REFERENCES

[1] M. Ainsworth, W. McLean and T. Tran, The conditioning of boundary element equations on locally refined meshes and preconditioning by diagonal scaling, *SIAM J. Numer. Anal.* **36** (1999), No. 6, 1901–1932.

[2] M. Ainsworth, W. McLean and T. Tran, Diagonal scaling of stiffness matrices in the Galerkin boundary element method, *ANZIAM J.* 42 (2000), 141–150.

[3] R. E. Bank and L. R. Scott, On the conditioning of finite element equations with highly refined meshes, *SIAM J. Numer. Anal.* **26** (1989), 1383–1394.

[4] M. Dauge, *Elliptic Boundary Value Problems on Corner Domains*, Lecture Notes in Mathematics, Springer-Verlag, 1988.

[5] J. Elschner, The double layer potential operator over polyhedral domains II: spline Galerkin methods, *Math. Math. Appl. Sci.* **15** (1992), 23–37.

[6] I.G. Graham, W. Hackbusch and S.A. Sauter, Hybrid Galerkin boundary elements on degenerate meshes. In: *Mathematical Aspects of Boundary Element Methods*, M. Bonnet, A.-M. Sändig and W.L. Wendland (eds.). CRC Press, 1999.

[7] I.G. Graham, W. Hackbusch and S.A. Sauter, Finite elements on degenerate meshes: Inverse-type inequalities and applications, *IMA Journal of Numerical Analysis* **25** (2005), 379-407.

[8] W. Hackbusch and Z. Nowak, On the fast matrix multiplication in the boundary element method by panel clustering, *Numer. Math.* **54** (1989) 463–491.

[9] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, 2000.

[10] J. C. Nédélec, Integral equations with non-integrable kernels, *Integral Equations Operator Theory* **4** (1982), 563–572.

[11] T. von Petersdorff, *Randwertprobleme der Elastizitätstheorie für Polyeder — Singularitäten und Approximation mit Randelementmethoden*, Dissertation, Technische Hochschule Darmstadt, 1989.

[12] V. Rokhlin, Rapid Solution of the Integral Equations of Classical Potential Theory, *J. Comput. Phys* **60** (1985), 187-207.