Thors Hans Hansson, March 10, 2025

QUANTUM FIELD THEORY IN FIVE EASY PIECES

CONTENTS

Introductory remark

These lectures aim to give you an idea of what quantum field theory (QFT) is. Why is it needed, what is its area of application, and why it is not only an essential tool for any theoretical physicist but also provides a way to understand how the physics of the microworld is connected to the phenomena observable in the laboratory and in ordinary life. I cover material that would usually take at least three full-length courses, so I have tried to scale down technicalities to a minimum and jumped many, many important and subtle point. However, I assume that you know undergraduate classical physics and quantum mechanics, so there will be equations, especially in the first lecture. So this is not "QFT for Dummies", but draws on your previous knowledge, by making analogies. My goal is that you will find the subject interesting and intellectually challenging, and will be motivated for serious studies of QFT.

Before starting, I will mention a couple of themes in the lectures:

I. There are two, to my knowledge equivalent, ways to construct a QFT

1. "Second quantization" where you start from many-body quantum theory and construct quantum field theory operators. This method is, or at least used to be, the preferred method in condensed matter (CM) physics, and is usually based on a *Hamiltonian* formulation of quantum mechanics.

2. "Field quantization" where a classical field theory is quantized, either using canonical, *i.e.* Hamiltonian, formulation or using path integrals that are based on Lagrangians. This method is the common one in high energy physics (HEP) and makes it easy to incorporate symmetries such as Lorentz invariance.

II. QM + relativity $\equiv$ relativistic QFT, which has some fundamental consequences that go beyond non-relativistic QM. Also QFT + gravity $\equiv$?? perhaps strings, perhaps loop quantum gravity, perhaps .....

III. Scale separation, Effective low-energy QFTs, and the connection between infrared and ultraviolet physics.

I. THE BASIC STRUCTURE OF QUANTUM FIELD THEORY

In this first lecture, I will tell you some basic features of QFT. How it describes the creation and annihilation of particles, how to describe interactions and how the set up a perturbation theory in terms of Feynman diagrams.

A. Annihilation and creation of particles – The Fock space

Ordinary quantum mechanics describes the dynamics of a fixed number of particles. This is perfectly adequate to describe atoms and molecules, and, with some effort, also solids and gases. There are, however, important phenomena that involve the creation and annihilation of particles. What comes first to mind might be the high-energy collisions in particle accelerators where many new particles are created. To understand such a process, one must find a way both to combine QM with special relativity and to find a formalism that allows for creation and annihilation of particles. Another important example is systems with gapless excitations such as photons or phonons. A third relevant example are the gapless excitations of quasiparticles and quasiholes at a Fermi surface of a metal.

In order to allow for quantum states with different numbers of particles, we must enlarge the Hilbert space to a *Fock space*

$$\mathcal{F} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots \,. \tag{1}$$

where $\mathcal{H}_n$ is an n-particle Hilbert space.

Now for this to be of any use, we must have operators that connect the different Hilbert spaces, *i.e.* operators that describe the *creation and annihilation of particles*. To see how to do this, we shall make a seemingly strange detour and remind ourselves of the one-dimensional quantum harmonic oscillator:

$$H = \frac{\hat{p}^2}{2m} + \frac{k}{2}\hat{x}^2 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \tag{2}$$

where $\omega = \sqrt{k/m}$ is the frequency of the oscillations, and the operators a and $a^\dagger$ satisfy,

$$[a, a^\dagger] = 1 \tag{3}$$

The quantum states and the corresponding energies are given by

$$|n\rangle = N(a^\dagger)^n\,|0\rangle \qquad , \qquad E_n = (n + \frac{1}{2})\hbar\omega\,, \tag{4}$$

where$N$ is a normalization constant.

Now think of this in another way. Instead of interpreting n as the excitation level of the oscillator, think of it as the number of particles. Then forgetting about the $\frac{1}{2}$ in (4) the energy is simply proportional to the number of particles, which is precisely what describes a system of non-interacting particles. We shall come back to the $\frac{1}{2}$ which has a very interesting interpretation in quantum field theory (QFT).

B. From the harmonic oscillator to a theory of photons

So knowing what to look for, can we find a quantum system that is described by a harmonic oscillator? As you will learn, there are many, but let us take one of the simplest, namely a resonance cavity that supports electromagnetic modes. The Hamiltonian is (in radiation gauge, $A_0 = 0$ and $\vec{\nabla} \cdot \hat{\vec{A}} = 0$; when you take a proper QFT course you will learn why I did not put a hat on A_0.)

$$H = \frac{1}{2} \int d^3r \, \left(\hat{E}^2(\vec{r}, t) + (\vec{\nabla} \times \hat{\vec{A}})^2(\vec{r}, t) \right) \tag{5}$$

where the *transverse fields* $\vec{A}$ and $\vec{E}$ are canonically conjugate. Just as for ordinary quantum mechanics, quantization corresponds to turning the Poisson brackets into commutator, and in this case, we get the the equal time commutation relations

$$[\hat{A}_i(\vec{r}_1, t), \hat{E}_j(\vec{r}_2, t)] = i\hbar \delta^{tr}_{ij}(\vec{r}_1 - \vec{r}_2) \tag{6}$$

where we use Heisenberg operators; from now on we shall omit the hats on the operators. So we can think of the electromagnetic field as *a quantum harmonic oscillator at each point in space* !! It is important that you realize that the analogy with the 1d harmonic oscillator is between the conjugate pairs (x, p) and A_i, E_j, while the $\vec{r}$ is just a label. The *quantum operators* A and E are examples of *quantum fields* and theories based on such operators are *quantum field theories* (QFT).

To connect more closely to the 1d h.o. example above, it is better to use a mode expansion of the em field,

$$\begin{aligned} \hat{\vec{A}}(\vec{r}, t) &= \frac{1}{2V} \sum_{a=1,2} \sum_{\vec{k}} \hat{A}_{a.\vec{k}} \, e^{-\frac{i}{\hbar}(\omega t - \vec{k} \cdot \vec{r})} \hat{\varepsilon}^a + c.c. \\ \hat{\vec{E}}(\vec{r}, t) &= \frac{1}{2V} \sum_{a=1,2} \sum_{\vec{k}} \hat{E}_{a,\vec{k}} \, e^{-\frac{i}{\hbar}(\omega t - \vec{k} \cdot \vec{r})} \hat{\varepsilon}^a + c.c. \end{aligned} \tag{7}$$

where $\hat{\varepsilon}^a$ are the two transverse polarization vectors satisfying $\vec{k} \cdot \hat{\varepsilon}^a = 0$, and V the volume of the volume If the system is defined in a box. In this case, the momenta $\vec{k}$ are discrete, while in open space they are continuous, and in this case the $\frac{1}{V} \sum_{\vec{k}}$ should be interpreted as a Fourier integral. Let us now for simplicity consider a single mode with polarization $\hat{\varepsilon}^a$ and momentum $\vec{k}$ and define

$$\hat{A}_{a.\vec{k}} = e^{-i\omega t} \hat{A}_{a.\vec{k}} \equiv \hat{A}(t) \, , \tag{8}$$

and similarly for $\hat{E}(t)$. A bit of algebra gives,

$$[\hat{A}(t), -\hat{E}(t)] = \frac{1}{V} i\hbar \quad . \tag{9}$$

Finally, we define

$$a(t) \equiv \sqrt{\frac{\omega V}{2\hbar}} \hat{A}(t) = \sqrt{\frac{\omega V}{2\hbar}} \left[\hat{A}(t) - \frac{i}{\omega} \hat{E}(t) \right] \tag{10}$$

$$a^\dagger(t) \equiv \sqrt{\frac{\omega V}{2\hbar}} \hat{A}^\dagger(t) = \sqrt{\frac{\omega V}{2\hbar}} \left[\hat{A}(t) + \frac{i}{\omega} \hat{E}(t) \right] \quad , \tag{11}$$

where the scaling factor between $a(t)$ and $\hat{A}(t)$ is chosen as to get the canonical commutation relation,

$$[a, a^\dagger] = 1\,. \tag{12}$$

Next, look at the Maxwell Hamiltonian,

$$\mathcal{E} = \int d^3x\, \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)\,, \tag{13}$$

which for our single mode takes the very simple form,

$$H = \frac{V}{2}\omega^2 \hat{A}^\dagger(t)\hat{A}(t) = \hbar\omega(a^\dagger a + \frac{1}{2})\,. \tag{14}$$

where we recalled that for a plane wave, $k^2 = \omega^2$.

It should now be clear that by starting from the vacuum state $|0\rangle$, we obtain n-photon states by repeated action of the creation operator,

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n\,|0\rangle \qquad ; \qquad E_n = \hbar\omega\left(n + \frac{1}{2}\right)\,. \tag{15}$$

where we reintroduced the normalization operator.

But how do we use these states to describe a classical electromagnetic wave? Again we can learn from the 1d harmonic oscillator, where the $|n\rangle$ states are very far from classical since $\langle n|\,\hat{x}\,|n\rangle = \langle n|\,\hat{p}\,|n\rangle = 0$ for all n. Instead, the *semi-classical states are the coherent states* defined by,

$$|\alpha\rangle = Ne^{\alpha a^\dagger}\,|0\rangle \tag{16}$$

where N is a normalization constant. These states satisfy $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$ and in such states, both $\hat{x}$ and $\hat{p}$ have well non-vanishing expectation values.

In analogy, we now form coherent states of the e.m. field by,

$$|\alpha_{a,\vec{k}}\rangle = Ne^{\alpha_{a,\vec{k}}\,a^\dagger_{a,\vec{k}}}\,|0\rangle \tag{17}$$

In later courses, you will learn that taking the interaction Hamiltonian,

$$H = \int d^3r\,\vec{J}(\vec{r},t)\cdot\hat{\vec{A}}(\vec{r},t) \tag{18}$$

where $\vec{J}(\vec{r},t)$ is a classical current, and then letting the corresponding time evolution operator,

$$U(T) = \mathcal{T}e^{-\frac{i}{\hbar}\int_0^T dt\,H(t)}\,, \tag{19}$$

where $\mathcal{T}$ is the time ordering operator, acts on the vacuum $|0\rangle$, will produce a coherent state that has the interpretation of a (semi)classical electromagnetic field.

The Fock space for the full theory is the product of the Fock spaces for the individual modes,

$$\mathcal{F}_{Max} = \prod_{a,\vec{k}} \otimes \mathcal{F}_{a,\vec{k}} \tag{20}$$

and the photon states are in obvious notation

$$|n_{a_1,\vec{k}_1}, n_{a_2,\vec{k}_1}, \cdots\rangle \tag{21}$$

i.e. we have $n_{a_1,\vec{k}_1}$ photons in the mode $(a_1, \vec{k}_1)$. Note that photons are bosons so we can have many particles in the same state. Later we will learn how to deal with fermions.

In the next section, we will generalize to the case where the current describes quantum particles. Again the interaction interaction term is,

$$\mathcal{L}_{int} = j_\mu A^\mu \tag{22}$$

but j_μ is now a quantum object that is constructed from quantum fields describing matter particles, that can be created and annihilated. In the next section, we shall learn how to construct such a current out of quantum fields that describe the creation and destruction of charged particles.

C. Lorentz invariant electrically charged matter

So now we have a theory for photons, but we also need a theory for matter. Since we want to describe the interaction with the quantized e.m. field, we must take charged matter particles. Electrons would be an obvious choice, but they will come later when we introduce fermions. Also, for simplicity, we will take bosons without spin. There are no known fundamental charged spinless bosons in nature (the famous Higgs boson which was discovered in 2012 at the LHC accelerator at CERN is neutral). There are, however, neutral, spinless, strongly interacting particles, such as the pions and kaons. We now know that these are bound states of quarks, but in many circumstances, as in traditional nuclear theory, they can be treated as elementary particles. Let us, however, for the time being, forget about the physical realization and assume that we have a spinless boson with mass m, and that we want to formulate a relativistic theory for such particles that allows for creation and annihilation. Before doing so, let's recall the two most important ways to approach quantum theory – the Hamiltonian and the Lagrangian formalisms.

A Hamiltonian quantum theory is defined by a Hamiltonian function $H(\hat{q}_1, \hat{p}_1, \ldots \hat{q}_n, \hat{p}_n)$ and a set of ETCs, $[\hat{q}_i, \hat{q}_j] = i\hbar$. Usually one identifies half of the variables $\hat{q}_i$ as momenta, but this is not necessary. The commutation relations define the Hilbert space of states, and the Hamilton function defines the time evolution, of these states.

A Lagrangian quantum theory is defined by a classical Lagrangian $L[q_1, \ldots q_n; \dot{q}_1, \ldots \dot{q}_n(t)]$, and a path integral defines the matrix elements of the evolution operator from $t = 0$ to $t = T$,

$$\langle q_i^{fi}| U(T) |q_i^{in}\rangle = \langle q_i^{fi}| \mathcal{T} e^{\frac{i}{\hbar}\int dt\, H(t)} |q_i^{in}\rangle = \int_{q_i^{in}(0)}^{q_i^{fi}(T)} \mathcal{D}[q_i(t)] e^{\frac{i}{\hbar} S[q_i(t)]}$$

where $S[q_i(t)] = \int_0^T dt\, L(q_i(t))$, and care must be taken to properly define the *functional integral measure* $\mathcal{D}[q_i(t)]$.

The Hamiltonian formulation is usually the better one to use when one is concerned with states, while the Lagrangian one is better for analyzing symmetries and also for extracting effective low-energy theories from high-energy microscopic models, an important theme in the following.

The Lagrangian approach is particularly useful for formulating Lorentz invariant field theories since it can easily be reformulated in an invariant way. To see this, we write the Lagrangian in terms of a Lagrangian density $\mathcal{L}$,

$$L(T) = \int d^d r \, \mathcal{L}[\phi(x), \partial_\mu \phi(x)] \tag{23}$$

where $x = (\vec{r}, t)$ is a space-time coordinate, so $S[\phi_i(x)] = \int_0^T dt \, \mathcal{L}(\phi_i(x))$. We also changed the notation from q to ϕ, which is a standard symbol for a *scalar field*.

We have already given the Hamiltonian version of the Maxwell quantum field theory that describes photons, and the corresponding Lorentz invariant Lagrangian is

$$\mathcal{L}_{Max} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \tag{24}$$

where $F_{\mu\nu}$ is the e.m. field tensor. Because of gauge invariance, complications occur if this is directly put into the path integral formula; you will learn about this when you take a proper QFT course. Here we shall, as already mentioned, take a scalar field which is much simpler to handle. For reasons that will become clear, we shall use a *complex scalar field* so that ϕ and $\phi^\star$ should be treated as independent variables.

The simplest Lorentz invariant Lagrangian density for such a field is,

$$\mathcal{L}_\phi = \frac{1}{2} \partial_\mu \phi^\star \partial^\mu \phi - \frac{m^2}{2} \phi^\star \phi \tag{25}$$

By construction, this is Lorentz invariant, and since it is quadratic in the complex field ϕ, it describes two scalar degrees of freedom in each point in space. From the corresponding action, we can by variation over $\phi^\star$ get the equations of motion,

$$(-\partial_\mu \partial^\mu - m^2)\phi(x) = 0 \tag{26}$$

or in energy-momentum space

$$(E^2 - k^2 - m^2)\phi_{\omega,\vec{k}} = 0 \,. \tag{27}$$

You should recognize the dispersion relation $E = \pm\sqrt{k^2 + m^2}$ as describing a massive relativistic particle. You might (or rather *should*) worry about the negative energies. We shall return to them when we discuss fermions, but here it suffices to say that they are related to *antiparticles* which, as we shall see below is an unavoidable consequence of a relativistic QFT.

To have Hamiltonian formalism, we introduce the *complex operator field* $\hat{\phi}$, which satisfies the equal time commutation relation (ECR),

$$[\phi(\vec{r}_1, t), \phi^\dagger(\vec{r}_2, t)] = i\hbar \delta^3(\vec{r}_1 - \vec{r}_2) \,. \tag{28}$$

This field can be Fourier expanded as,

$$\hat{\phi}(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}\left(a_{\vec{k}}\, e^{-ik\cdot x} + b^\dagger_{\vec{k}}\, e^{ik\cdot x}\right) . \tag{29}$$

where $E_{\vec{k}} = \sqrt{k^2+m^2}$, and where we note that for a real field, we must have $b^\dagger_{\vec{k}} = a^\dagger_{\vec{k}}$. The physical interpretation is that the positive frequency component $\sim a_{\vec{k}}$ destroys a particle, while the negative frequency component $\sim b^\dagger_{\vec{k}}$ creates an antiparticle. More on this below. For a real field, like a neutral pions, the particle would be its own antiparticle, but for the complex field, they are distinct and differ by the sign of the charge. (This charge is not necessarily the electric charge but is a quantum number that distinguishes particles from antiparticles.)

The reason for charged particles being described by complex quantum fields is similar to why you need a complex wave function to couple to electromagnetism. To see this, recall that the Schrödinger current is given by,

$$\vec{j} = \frac{\hbar}{2mi}\left(\psi^\star\vec{\nabla}\psi - \psi\vec{\nabla}\psi^\star\right) \tag{30}$$

which vanishes if ψ is real.

A technical comment

The reason for the factor $\frac{1}{2E_{\vec{k}}}$ in (29) is Lorentz invariance, as can be seen by,

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_{\vec{k}}} f(\vec{k},k_0) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2-m^2) f(\vec{k},k_0) \tag{31}$$

which holds for any function $f(\vec{k},k_0)$. This shows that the Fourier transformation respects Lorentz invariance. A consequence is that the canonical commutation relations that follow from (28) read,

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = 2E_{\vec{k}}(2\pi)^3\delta^3(\vec{k}-\vec{k}') \qquad \text{etc.} \tag{32}$$

D. Propagation of particles

With this, we have everything that is needed to describe the propagation of charged massive particles, but for what follows next, you should recall yet another result from non-relativistic QM (see *e.g.* chap. 2.5 in Ref. **?**),

$$K(x,t;x_0,t_0) = \langle x|\, U(t-t_0)\,|x_0\rangle = \langle x,t|x_0,t_0\rangle\,. \tag{33}$$

The function, K is the non-relativistic *propagator* which is the amplitude for finding a particle at position x at time t given that it was at position x_0 at time t_0. We also have the boundary condition $K(x,t;x_0,t_0) = 0$ for $t < t_0$ meaning that nothing can propagate backward in time. The last expression in (33) is in terms of Heisenberg states, where $|x,t\rangle$ is an eigenstate of the Heisenberg operator $x(t)$. In analogy, we now define a *QFT propagator*, as

$$G(\vec{r}_2,t_2;\vec{r}_1,t_1) = \langle 0|\, T[\phi(\vec{r},t)\phi^\dagger(\vec{r}_0,t_0)]\,|0\rangle \tag{34}$$

where the symbol T is a time-ordering meaning,

$$T[A(t_2)B(t_1)] = \theta(t_2 - t_1)A(t_2)B(t_1) + \theta(t_1 - t_2)A(t_2)B(t_2) \tag{35}$$

This time ordering has a very interesting interpretation. First note that for the non-relativistic propagator only the first term, with the ordering $\phi(\vec{r}, t)\phi^\dagger(\vec{r}_0, t_0)$, is present, and this we already interpreted as creating a particle at $(\vec{r}_0, t_0)$ and destroying it later at $(\vec{r}, t)$. This is possible since in non-relativistic physics the notion of "before" and "after" has an absolute meaning, which is not the case in a relativistic theory. The other time ordering, $\phi^\dagger(\vec{r}, t)\phi(\vec{r}_0, t_0)$, which does not seem to make any sense, is something created before it is destroyed?? The resolution to this conundrum is that you should not think of $\phi^\dagger$ as creating a particle, but as creating a charge, and creating a positive charge can be done either by creating it directly from the vacuum or by destroying an already exciting particle with a negative charge. Such particles are called *antiparticles* and arise with necessity in relativistic QFT. They do not have to be charged, the essential point is that a particle and an antiparticle can annihilate each other.

Since the antiparticle is described by the same theory, it has the same mass (and also the same spin, but this comes later) as the particle. Sometimes the time ordering $\theta(t_1 - t_2)$ is thought of as a particle propagating backward in time, but that usually leads to confusion. You should think of it as an antiparticle propagating forward in time. You should as an exercise insert the Fourier expansion (29) into (34) to see that this discussion is consistent with the interpretation of the operators $a_{\vec{k}}$ and $b_{\vec{k}}$ that we gave earlier.

The QFT propagator $G(\vec{r}_2, t_2; \vec{r}_1, t_1)$ describes a propagation from one space-time point to another, but another very important object is its Fourier transform, $G(k)$ defined by,

$$G(k) = \int d^4x \, e^{ik_\mu x^\mu} G(x, 0) \tag{36}$$

where $k^\mu = (\omega, \vec{k})$ is the momentum 4-vector. $G(k)$ describes the propagation of a free particle, that is a particle with a fixed 4-momentum. In a typical experiment, one sends in particles with some specified momenta, and after a scattering process measures the momenta of the outgoing particles, $G(k)$ is a very useful object that we will come back to below.

In section 42 I will outline how to calculate the propagator $G(k)$ using the Lagrangian (25), and the path integral formula for fields. It turns out that it is simply the inverse of the kernel of the quadratic part of the Lagrangian,

$$G(k) = \frac{i}{k^2 - m^2 + i\epsilon} \tag{37}$$

where the term $i\epsilon$ is crucial in order to properly define what is happening "on shell" that is when $k^2 = m^2$ which is the dispersion relation for a freely propagating particle. We shall not further dwell on these (very important) technicalities.

E. Interactions between matter and radiation – Feynman diagrams

We already mentioned that QFT allows for the creation and destruction of particles. Now we give examples of such processes and outline how they are described mathematically.

The perhaps most familiar example is the emission of photons from an accelerated charge. This is described in classical e.m. theory, as an emission of electromagnetic waves, but as we saw above, such states can also be thought of as coherent states of photons. In addition, with modern technology, one can construct sources that emit a single photon. Such a process is written as,

$$|\vec{k},\pm\rangle \to |\vec{k}'\pm\rangle + \gamma \tag{38}$$

another process is the annihilation of a charged particle and its antiparticle into a photon,

$$\gamma \to |\vec{k}+\rangle + |\vec{k}'-\rangle \tag{39}$$

where the sign denotes the charge of the particles. These processes can also be described graphically, by the *Feynman diagrams*, in Fig. 1a and b.

The external lines in these pictures represent incoming or outgoing photons and charged scalar particles with some given momenta, while the internal lines represent propagators, $G(k)$ and $G_{\mu\nu}(k)$ for scalars and photons respectively. The latter is given by a formula very similar to (37).

The vertices represent the coupling term (22), where the current must be some combination of the fields ϕ, $\phi^\dagger$ that create and annihilate particles. So we must construct a current 4-vector from the operators, and since this current should be an observable it must be hermitian, and furthermore gauge invariant. Drawing inspiration from (30) we have

$$j_\mu = \phi^\dagger(-i\partial_\mu - eA_\mu)\phi\,, \tag{40}$$

below I will indicate how you can derive this expression from the interaction Lagrangian. With this, the interaction Lagrangian becomes

$$\mathcal{L}_{int} = A^\mu\phi^\dagger(-i\partial_\mu - eA_\mu)\phi\,. \tag{41}$$

and you should note that this term describes the creation and/or annihilation of particles, antiparticles, and photons, in such a way that both 4-momentum and electric charge is conserved.

In Fig. 1a, a particle with momentum $\vec{k}$ is destroyed and a particle with momentum $\vec{k}'$ is created while at the same time creating a photon. The photon carries away the missing 4-momentum $\vec{k}-\vec{k}'$ so that total energy and momentum is conserved. The rule of the game is that both energy and momentum is conserved in every vertex.

This is a good place to introduce the very important concept, of *off-shell* particles. In the diagrams in Fig. 1, the external lines correspond to freely moving relativistic particles, so the energies and momenta must satisfy $E^2 = \vec{p}\cdot\vec{p}$ or $p^2 = 0$. For the internal lines in Figs. 1c and 1d this is not the case. These

particles only "exist" for a short time when the interaction takes place, and we call them *off-shell* or *virtual* particles. The particles described by the external lines are called *on-shall* or *real*.

A technical comment
I cheated a little bit about the diagrams. Doing things carefully you find that there are extra pieces corresponding to diagrams where two photons are emitted/absorbed from the same space-time point. Try to figure out why such terms are called "seagull diagrams". Below I will retain this contribution.

F. More on Feynman diagrams – loops

With this as a starting point, we can go on drawing pictures. The two diagrams in Fig.1e and 1f, differ from the others in that they contain a loop. A heuristic way to think about 1e is that a photon transforms into a particle-antiparticle pair, which is then annihilated into a photon. Since energy and momentum are conserved the outgoing photon must have the same four-momentum as the incoming one. The momenta of the virtual particles in the loop are not fixed, so the mathematical expression for such a diagram includes an integral over the momentum in the loop. You should now be able to make a similar interpretation of the diagram 1f.

But you might wonder what all these pictures mean. The answer is that they are a very convenient way to represent a *perturbation expansion*. The propagator lines correspond to the unperturbed Hamiltonian H_0 (or Lagrangian L) that describes free particles, and the vertices represent the interaction Hamiltonian H_I. Given these Hamiltonians, one can deduce what are the propagators and the vertices. For a given process, the external lines corresponding to incoming particles are usually fixed and depend on the experimental setup. The external lines corresponding to outgoing particles are usually not fixed but should be integrated over some range of momenta determined by the detectors. The internal *loop momenta* are integrated over. In coming courses, you will learn how to go from the Hamiltonian or Lagrangian describing the theory to the *Feynman rules* that tell you how to draw and evaluate diagrams. Each vertex comes with a coupling strength so when you add more internal lines you get higher and higher order terms in the perturbation expansion. Although this looks quite different from the perturbation expansions you have met in non-relativistic quantum mechanics, it is quite similar. Recall that in this case you have energy denominators and sums over intermediate states. In the relativistic perturbation expansion based on Feynman diagrams, the sums over intermediate states translate into the integrations over loop momenta. Even in this case, one can use "old-fashioned" or "time-ordered" perturbation theory with an energy nominator, but this is usually much more complicated.

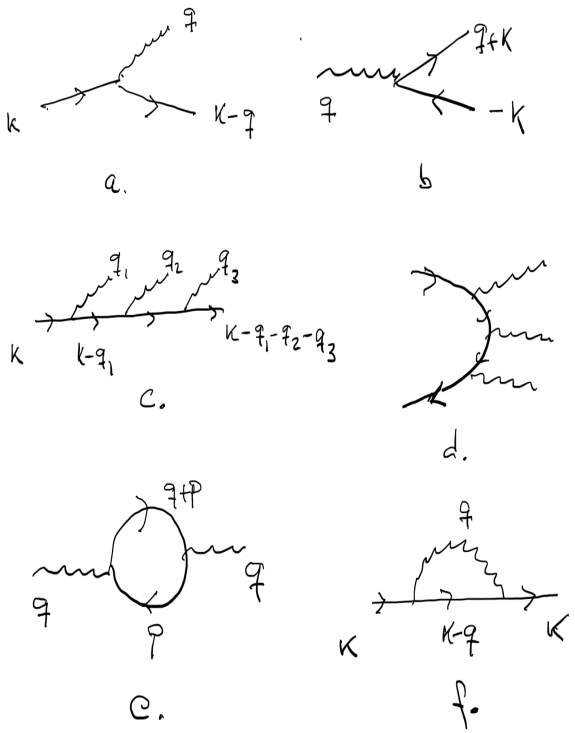

Figure 1 Feynman diagrams. a. emission of a photon from an electron or positron. b. annihilation of an particle-antiparticle pair into a photon. c. emission of several photons from a particle or antiparticle, the internal lines are off-shell. e. particle-antiparticle annihilation into several photons. e. 1-loop correction to the photon propagator due to the creation of electron-positron pairs. f. 1-loop correction to the particle propagator due to emission and reabsorption of a photon.

G. Quantum Field path integrals

I will now outline how perturbation theory is done mathematically, and we start by showing how the propagator $G(k)$ is obtained from a field theory path integral.

One can calculate $G(k)$ by evaluating the expectation value in (34) and then take the Fourier transform, but a simpler way is to use the QFT version of the path integral formula (23),

$$\langle \phi(T,\vec{r}_2|\, U(T)\, |\phi(0,\vec{r}_1)\rangle = \int_{\phi(0,\vec{r}_1)}^{\phi(T,\vec{r}_2)} \mathcal{D}[\phi(t,\vec{r})]\, e^{\frac{i}{\hbar}S[\phi]} \tag{42}$$

Note that $\vec{r}$ is an index just as i was in (23), which means that S is a *functional* of the fields. Otherwise the interpretation is very similar, it describes the time evolution of a QFT state at $t = 0$ to to another at $t = T$.

Of special interest in high-energy physics is the *scattering operator* $S = U(\infty, -\infty)$, For a complex scalar field coupled to electromagnetism, this is,

$$S = \int_{\phi,\phi^\star,A_\mu(-\infty,\vec{r})}^{\phi,\phi^\star,A_\mu(\infty,\vec{r})} \mathcal{D}[\phi(t,\vec{r})]\mathcal{D}[\phi^\star(t,\vec{r})]\mathcal{D}[A_\mu(t,\vec{)}]\, e^{\frac{i}{\hbar}S[\phi,\phi^\star,A_\mu]} \tag{43}$$

where we included a gauge invariant coupling to the electromagnetic field by taking

$$S[\phi,\phi^\star,A_\mu]=\int d^4x\,\left[\phi^\star(-D_\mu D^\mu-m^2)\phi(x)-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right] \tag{44}$$

with $iD_\mu=i\partial_\mu+eA_\mu$. This is an unphysical limit, but appropriate for calculating the scattering events that are measured in particle physics accelerators.

Since these states can be described as a set of particles, this time evolution can be thought of as a number of world lines, but note that the number of particles is in general not conserved during this time evolution. From the matter part of (44) you can derive the expression (40) for the current using the relation,

$$j_\mu(x)=\frac{\delta S[A]}{\delta A^\mu(x)}\,. \tag{45}$$

The integral in (43) is very complicated, so we first look at a simpler version where the electromagnetic field is not fluctuating, i.e. it does not have any dynamics of its own. Then,

$$S=U(-\infty,\infty)=\int_{\phi,\phi^\star(-\infty,\vec{r})}^{\phi,\phi^\star(\infty,\vec{r})}\mathcal{D}[\phi(t,\vec{r})]\mathcal{D}[\phi^\star(t,\vec{r})]\,e^{\frac{i}{\hbar}S[\phi,\phi^\star,\bar{A}_\mu]} \tag{46}$$

where $\bar{A}$ is non-dynamical. This is enough to describe how electrons are deflected by an external field and how electron-positron pairs are produced in very strong electric fields.

To carry out the integral in (46), we rewrite the action as

$$S[\phi,\phi^\star,\bar{A}_\mu]=\frac{1}{2}\int d^4x\,\left[\phi^\star(-\partial_\mu\partial^\mu-m^2)\phi(x)-2e\phi^\star\partial^\mu\phi\bar{A}_\mu(x)+e^2\phi^\star\phi\bar{A}_\mu\bar{A}^\mu\right]\,, \tag{47}$$

and then (46) can be taken as the starting point for a perturbation expansion in the electric charge e, where we recall that $e^2/2\pi=\alpha\approx 1/147$. (Fill in factors of $\hbar$ and c for yourselves!) In such an expansion, each field either corresponds to an external particle or an internal line where the endpoints are integrated over. All we have to do is to perform the integrals in (46), which is simple since they are Gaussian. We must, however, remember that there is one integration variable at each space-time point, so we need the formula,

$$\int\prod dx_i e^{-\frac{1}{2}\sum_{i,j}x_iM_{i,j}x_j-j_ix_i}\sim\frac{1}{\sqrt{\det(M)}}e^{-\frac{1}{2}\sum_{i,j}j_iM^{-1}_{i,j}j_j} \tag{48}$$

In the case of complex variables, as in (46) there are two real variables, so there is no square root in the formula. We need yet another formula, or rather trick, for evaluating integrals,

$$\begin{aligned}\int dx\,e^{-\frac{k}{2}x^2+\lambda f(x)} &= \left[1+\lambda f\left(\frac{d}{da}\right)+\frac{\lambda^2}{2}f^2\left(\frac{d}{da}\right)\ldots\right]\int dx\,e^{-\frac{k}{2}x^2+ax}|_{a=0} \\ &\sim \left[1+\lambda f\left(\frac{d}{da}\right)+\frac{\lambda^2}{2}f^2\left(\frac{d}{da}\right)\ldots\right]e^{-\frac{1}{2}a\frac{1}{k}a}\,.\end{aligned} \tag{49}$$

If we now rewrite (47) as

$$S[\phi,\phi^\star,\bar{A}_\mu]=\frac{1}{2}\int d^4x\,\left[\phi^\star G^{-1}\phi(x)+\mathcal{L}_{int}(\phi^\star,\phi,A_\mu)-j^\star\phi-j\phi^\star\right] \tag{50}$$

where we introduced a set of auxiliary variables $j(x)$ and $j^\star(x)$, we can use (48) and (49) to get,

$$S=\left(1+S_{int}+\frac{1}{2}S_{int}S_{int}+\ldots\right)e^{\frac{i}{2}\int d^4xd^4y\,j(x)G(x-y)j(y)} \tag{51}$$

where $G(x)$ is the Fourier transform of G(k) and,

$$\hat{S}_{int} = \int d^4x\, \mathcal{L}_{int}\left(\frac{\delta}{\delta j(x)}, \frac{\delta}{\delta j^\star(x)}, A_\mu(x)\right) . \tag{52}$$

Now you can see how things work. Taking S_{int} from (47) and letting it act once on the exponential in (51) you get a term,

$$\sim 2e \int d^4y \int d^4z \int d^4x \bar{A}_\mu j^\star(y) G(y-x) i\partial^\mu G(x-z) j(z)\,. \tag{53}$$

Acting again with S_{int} will result in longer strings with convoluted propagators attached to $\bar{A}$ factors representing the external classical field. Recalling that convolutions in x-space gives multiplication in momentum space you can see how Feynamn grahps are starting to emerge. In the full theory, A_μ is also dynamical and have a propagator, and including this will give the full theory of Quantum Electro-Dynamics (QED).

There are lots of technical issues that I have glossed over, related to time ordering and normalizations and how to handle the "external lines" that describe the incoming particles. It turns out that for these lines, the propagators are canceled and one gets what is called amputated graps. The reason, I did not show you how to obtain the full theory by also integrating over the photon field A_μ, is that it is technically more difficult due to the gauge invariance of the Maxwell action. However, by doing this carefully you will get an expression for the full photon propagator which is the internal wavy line in Fig. 1f. In spite of skipping all these details, you have hopefully, got a basic understanding of what Feynman diagrams are about beyond just pretty pictures.

H. More on loop diagrams – Renormalization

With this in hand, we can now look a bit more closely to the loop diagrams in Fig. e and f. I both cases there is a momentum in loop that is not fixed by external momenta, but must be integrated over. Here I will jump a bit ahead and give the results for the case where the charged particles are fermions, which is the relevant case for QED.

In the second diagram, called a self-energy digram since it gives a correction to the mass of the particle, the integral is schematically (i.e. by counting powers of the loop momenta)

$$\Sigma(k) \sim e^2 \int d^4q\, \frac{q(k-q)}{q^2(k-q)^2} = \frac{3\alpha}{\pi} m \ln\frac{\Lambda^2}{m^2} + \dots\,, \tag{54}$$

where $\alpha = e^2/4\pi \approx 1/137$ is the fine structure constant, all Lorentz indicies and gamma matrices are suppressed, and the momenta in the numerator comes from the derivatives at the vertices. This integral is superficially linearly divergent at large p, but the linear terms $\sim m$ and $\sim k$ vanish due to a symmetric integration over p. The remaining part is logaritmically divergent, so if we put a cutoff $|p| < \Lambda$, it will contain a term $\sim \ln\Lambda$. The divergent piece can be incorporated into a renormalization of the electron mass m_e.

In the first diagram, the integral looks like

$$\Pi(q)_{\mu\nu} \sim e^2 \int d^4p \, \frac{q(k-q)}{q^2(k-q)^2} = (k^2\delta_{\mu\nu} - k_\mu k_\nu)(\frac{\alpha}{3\pi} \ln \frac{\Lambda^2}{m^2} + \dots) \,, \tag{55}$$

again omitting all details in the middle step. Here the divergence is again only logarithmic, this time because the prefactor of dimension two that depends on the external momenta. This tensor structure is guaranteed by gauge invariance. Putting back the gauge potentials we get

$$A^\mu \Pi(q)_{\mu\nu} A^\nu = F_{\mu\nu} F^{\mu\nu} (\frac{\alpha}{3\pi} \ln \frac{\Lambda^2}{m^2} + \dots) \tag{56}$$

which corresponds to a renormalization of the Maxwell action. By rescaling the potential A_μ this effectively amounts to a renormalization of the electric charge e.

You might think that the above procedure is just sweeping the problem under the carpet, and in this you are not alone. When Feynman, Schwinger, and Dyson came up with this idea of "renormalization" many physicists were quite unhappy, with the prescription

$$e_{ren} = \left(1 + \frac{\alpha}{3\pi} \ln \frac{\Lambda^2}{m^2}\right) e_{bare} \tag{57}$$

$$m_{ren} = \left(1 + \frac{3\alpha}{\pi} \ln \frac{\Lambda^2}{m^2}\right) m_{bare} \,. \tag{58}$$

where e_r and m_r are identified as the physically measured quantities that will appear in all results for observables, like cross sections or decay rates. (To be precise, in a modern version of this procedure things are done somewhat differently using the concept of counterterms.) What makes this, seemingly arbitrary and ad hoc procedure, interesting and successful is that you can extend this procedure to higher order in perturbation theory and absorb *all divergences in all diagrams* by a suitable renormalization of the bare parameters. Such theories are called renormalizable and have much more predictive power than non-renormalizable theories where one has to introduce new parameters in every order of the perturbative expansion. The proofs that QED, and later the standard model, are renormalizable were some of the great theoretical achievements of early-day QFT. In the third lecture we shall return to the question of renormalizability, and introduce a newer, and by now generally accepted, way to understand what it means.

II. IDENTICAL PARTICLES – A DEEP CONCEPT

The notion of identical particles is a fundamentally quantum mechanical concept since in classical physics one can always imagine "labeling" particles in a way that does not change the dynamics. You also know from non-relativistic QM that elementary particles can be divided into two groups – fermions and bosons, depending on whether the many-body wave functions are totally antisymmetric or totally symmetric respectively. In both relativistic and non-relativistic QFT, the quantum statistics of bosons is a consequence of the commutation relations like those in (28) that imply,

$$|\vec{r}_1, \vec{r}_2\rangle = \phi^\dagger(\vec{r}_1)\phi^\dagger(\vec{r}_2)\,|0\rangle = \phi^\dagger(\vec{r}_2)\phi^\dagger(\vec{r}_1)\,|0\rangle = |\vec{r}_2, \vec{r}_1\rangle\ . \tag{59}$$

You might also know that in order to describe fermions, which have fully antisymmetric states, we must demand that the corresponding creation and annihilation operators *anti-commute*,

$$\{\psi(\vec{r}_1, t), \psi^\dagger(\vec{r}_2, t)\} = i\hbar\delta^3(\vec{r}_1 - \vec{r}_2)\,. \tag{60}$$

where $\{A, B\} = AB - BA$, and we follow the usual convention and denote fermionic fields with ψ. With this the fermionic version of (59) becomes,

$$|\vec{r}_1, \vec{r}_2\rangle = \psi^\dagger(\vec{r}_1)\psi^\dagger(\vec{r}_2)\,|0\rangle = -\psi^\dagger(\vec{r}_2)\psi^\dagger(\vec{r}_1)\,|0\rangle = -\,|\vec{r}_2, \vec{r}_1\rangle\ . \tag{61}$$

In the rest of this section, we shall address three questions:

1. How does quantum statistics enter in the quantum mechanical path integral formalism?
2. Are there any other types of particles except fermions and bosons?
3. How to write a QFT path integral for fermions?

A. Quantum statistics and the path integral

In QM, the time evolution operator (23) is a sum over particle trajectories weighted with a phase given by their action. For a state with more than one particle, the recipe is the same, just with one trajectory for each particle, as illustrated in Fig. 2. Remember that in QM the number of particles is unchanged under time evolution.

With this, we can now understand how Bose statistics is manifested in the path integral. That the particles are indistinguishable, means that two states that only differ by the labeling of the particles are in fact identical. This means that paths that differ only by the permutation of the endpoints (and/or the starting points) are identical and should be summed over, see Fig. 2. Fynman pointed out, that when defining the path integral, one has the freedom to assign an arbitrary relative phase to trajectories that cannot be continuously deformed into each other. However, there is a severe constraint on how these relative phases can be assigned: the time evolution operator must obey the composition rule:

$$U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)\,. \tag{62}$$

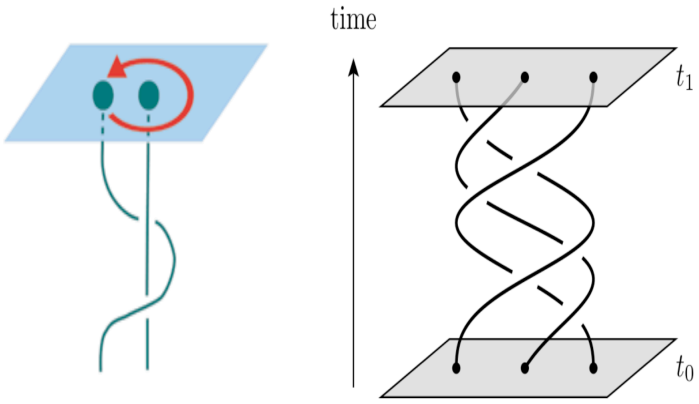

Figure 2 Trajectories of identical particles in 2+1 dimension. The left panel shows how the braiding of two worldlines describes one particle circling another. The right panel shows a more general braiding of three particles.

Some reflection gives that this is true if we for each set of trajectories assign either 1 or -1, depending on whether the permutation is even or odd. This corresponds to bosons and fermions respectively. Since any permutation can be obtained by a series of exchanges, this is really nothing but the standard argument for quantum statistics based on wave functions. Since wave functions are defined only up to a constant phase, we have for two particles: d where α is the exchange phase. From this follows that $e^{i\alpha} = \pm 1$, or $\alpha = 0$ or $\pi \mod(2\pi)$.

B. Fermions, Bosons and Anyons

However, looking at the pictures in Fig. 2, it looks as if we could assign a phase $e^{2i\alpha}$ for a "braiding" that does not exchange any particles. So why not assign a phase $e^{i\alpha}$ with a general α for an elementary exchange, since the trajectories corresponding to different braids cannot be deformed into each other? You are, however, somewhat fooled by looking at pictures, since these describe trajectories in (2+1) D space-time. In (3+1)D there are no braids! But this has very interesting consequences. In (2+1) D space-time there *are* alternatives to bosons and fermions, and particles with $\alpha \neq 0, \pi$ are called anyons. In a later section, you will learn about a physical system where such particles indeed exist. They were experimentally discovered

only a couple of years ago, although they were theoretically predicted much earlier.

C. QFT path integral for fermions

Now to fermions that seems to pose a real problem for the QFT path integral. This is illustrated by going back to ordinary QM and deriving the path integral formula by inserting complete sets of coherent states of the type (16) in a transition amplitude using the resolution of unity,

$$1 = \int \frac{d\alpha d\overline{\alpha}}{\pi} |\alpha\rangle \langle\alpha| \tag{63}$$

where we recall that $a|\alpha\rangle = \alpha|\alpha\rangle$. The corresponding equation for a fermionic operator c would be $c|\xi\rangle = \xi|\xi\rangle$, but since $cc = 0$ by anticommutation, we get $\xi^2 = 0$. This means first that there are only two states in the coherent superposition,

$$|\xi\rangle = e^{\xi c^\dagger} = |0\rangle + \xi|1\rangle \tag{64}$$

and secondly, that the parameter ξ in the *fermionic coherent state* $|\xi\rangle$ is a *Grassmann number*. We also need the "complex conjugate". $\bar{\xi}$, but it is important to remember that ξ and $\bar{\xi}$ should be treated as independent variables, A set of such numbers obeys,

$$\{\xi_i, \xi_j\} = \{\bar{\xi}_i, \bar{\xi}_j\} = \{\bar{\xi}_i, \xi_j\} = 0\,. \tag{65}$$

Just as real numbers, the Grassmann numbers can be real or complex,

$$\xi = \gamma_1 + i\gamma_2 \tag{66}$$

$$\overline{\xi} = \gamma_1 - i\gamma_2\,, \tag{67}$$

for most of the time, we shall only deal with the complex Grassmann numbers, but at the very end, we might say something about the "real" γ_i's that turn out to be related to quantum computing!

A product of an even number of Grassmann numbers is an ordinary number, just as an even number of anticommuting fermion operators, c and $c^\dagger$ is a boson operator. With this, we are ready to introduce fermionic, or anticommuting, field operators $\hat{\psi}(t, \vec{r})$ that satisfy the following equal time anti-commutation relations,

$$\{\psi(t, \vec{r}_1), \psi(t, \vec{r}_2)\} = 0 \tag{68}$$

These can be combined to form bosonic fields, e.g. $\hat{\rho}(x) = \psi^\dagger(x)\psi(x)$ is the density operator, and the corresponding classical object that can be used in a path integral is $\rho(x) = \bar{\psi}(x)\psi(x)$, where $\psi(x)$ and $\bar{\psi}(x)$ are Grassmann-valued fields, i.e. a Grassmann number defined in each space-time point.

The next step is to find an action, and since we can now describe fermions, the obvious goal is to find an action for electrons. Not surprisingly, this will be directly related to the Dirac equation. To see this, take the action,

$$S_D[\bar{\psi}, \psi] = \int d^4x\, \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{69}$$

where as usual $\not{p} = \gamma^\mu p_\mu$, where the 4-dimensional matrices γ^μ satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Taking the (functional) derivative with respect to $\bar{\psi}$ yields the Dirac equation $(i\not{\partial} - m)\psi = 0$. (The technical details concerning taking derivatives with respect to Grassmann numbers, you will learn in regular QFT courses.)

Coupling to electromagnetism is as usual by making the minimal substitution $i\partial_\mu \to i\partial_\mu - eA_\mu$. Combining the resulting action $S_D[\bar{\psi}, \psi, A]$ with the Maxwell action as in (44), we have the full theory of Quantum ElectroDynamics – QED!

To calculate in this theory in the same way as we did for the bosons described by the scalar field ϕ, we must learn how to integrate over Grassmann variables. The only rules you need for this is,

$$\int d\xi = 0 \qquad ; \qquad \int d\xi\, \xi = 1\,, \tag{70}$$

and similarly for $\bar{\xi}$. Note that there are no limits in these integrals, and you should think of them as purely formal recipes. Using this rule you can figure out the Grassmann version of (48),

$$\int \prod_i d\xi_i d\bar{\xi}\, e^{-\sum_{i,j} \bar{\xi}_i M_{i,j} \xi_j - \bar{\chi}_i \xi_i - \bar{\xi}_i \chi} = \det(M) e^{-\sum_{i,j} \bar{\chi}_i M^{-1}_{i,j} \chi_j} \tag{71}$$

where χ and $\bar{\chi}$ auxiliary Grassmann fields playing the same role as j in (48). By now, you should be familiar with how to go from integrals over functions of many variables, to functional integrals over fields,

The fermionic propagator, corresponding to the bosonic $G(k)$ is

$$S(k) = \frac{1}{\not{k} - m} = \frac{\not{k} + m}{k^2 - m^2} \tag{72}$$

and such a factor will be included for each internal electron line in a Feynman graph. In case you are not familiar with the notation $\not{k} = \gamma_\mu k^\mu$, just notice how the integrand scales with k at high momenta.

We end this section with two comments.

A. QED, which is part of the standard model of elementary particles, is one of the most successful theories in all of physics. It explains a large number of precision experiments and forms the theoretical basis for a precise understanding not only of atomic and condensed matter physics but also of chemistry.

B. In non-relativistic QM the main object is traditionally the Hamiltonian and the corresponding Schrödinger equation determining the wavefunction, but you know that there is also a Lagrangian formulation. So far we have mainly discussed the Lagrangian formulation of QFTs such as QED. But is there a Hamiltonian formulation? The answer is yes, and there is indeed also a Schrödinger *function equation* that looks precisely as the one are used to. In the case of QED,

$$H[\hat{\phi}^\dagger, \hat{\phi}, \hat{A}_\mu]\Psi[\phi^\star, \phi, A_\mu] = i\hbar \frac{\partial}{\partial t} \Psi[\phi^\star, \phi, A_\mu]\,. \tag{73}$$

Note that although the field equations for the operator ψ e.t.c., are in general non-linear, the Schrödinger functional equation is linear, as demanded by probability conservation.

III. HOW CAN THE WORLD BE INTELLIGIBLE?

A very natural question when studying physics, is how we can at all understand matter at large scales without understanding the underlaying "substructure". We know that this is possible. Newton did not know about atoms, and Maxwell did not know about photons. Before the 1960'ies protons and pions were assumed to be "elementary" while we now know that they are made of quarks and gluons.

The resolution to this puzzle lies in some key concepts, the most important ones being symmetry and scale separation. The latter means that to describe phenomena at long times and large distances, or equivalently at low energy and small momenta, the physics at small scales can be captured by a few parameters, such as mass, charge and magnetic moment of different particles. As a concrete example, to do atomic physics, you do not have to know about nuclear structure, but just know the mass and charge of the nucleus. For very heavy atoms, you might also want to know the radius of the nucleus. Scale separation thus allows us to formulate *effective low energy theories*. In fact, for all we know, all our experimentally verified theories, such as the standard model of elementary particles, are most likely effective theories, and elementary particle physics is all about finding a microscopic basis for it. From this it should also be clear that what is meant with word like "short", "low" etc. depends on what kind of phenomena you are studying. For a cosmologist, a light year is "short" and for a string theorist a femtometer is "long".

So the next question is how to construct effective theories and models. Here symmetry plays an important role. The symmetries of nature is at the end a matter of observation and experiments. As far as we know, the symmetries of space and time are well captured by the special and general theory of relativity. In all cases we are concerned with, special relativity will do, and in most cases even Galilean symmetry. These symmetries are related to conservation laws, as you know from classical mechanics and electromagnetism. Then there are other, so-called internal symmetries, such as isospin symmetry and color symmetry, that are related to the conservation of various kinds of charges. The presence of these symmetries are in fact deduced from the conservation laws, while some of the space time symmetries were empirically established before the connection to the symmetries was established. Some symmetries, as for instance isospin, are only approximate, and we call them broken symmetries. They are nevertheless often important in constructing models and theories. A special, and very important kind of symmetry breaking is the one that occurs when the Hamiltonian, or Lagrangian, is fully symmetric, but the ground state does not respect the symmetry. A simple example is a crystal. The interactions between atoms are all translationally invariant, but a crystal is not – the continuous translational symmetry is broken to translations by lattice vectors. This phenomenon is called *spontaneous symmetry breaking* (SSB) and will be important in the following.

A. Effective Quantum Field Theories

Quantum field theory provides a systematic way to construct low energy theories. The basic idea is to "integrate out" high energy degrees of freedom (d.o.f.) to be left with a theory that only depends on low

energy d.o.f. Starting from an action $S_0[\psi_l, \psi_h]$ this procedure is schematically captured by,

$$e^{iS_{eff}[\phi_l]} = \int \mathcal{D}[\phi_h] e^{iS_0[\psi_l, \psi_h]} \tag{74}$$

where h and l stand for high and low. How to define the split, and how to perform the integral is in general a very hard problem, but we shall later consider a typical case. There are three related, but conceptually distinct, way to isolate the low-energy d.o.f. that we now describe.

1. Effective response actions

The simplest is to consider a microscopic quantum system coupled to some external field, typically electromagnetism, but stress or strain are other possibilities. Assuming the e.m. fields to be at low $(\omega, \vec{q})$ we can integrate out the fields describing the matter, to get an *effective response action*, $S_{eff}[A_\mu]$ that can be used to calculate response functions such as conductivities.

2. Effective actions for composite fields

Another way to define low energy variables is to form *composite fields*. The archetypical example in condensed matter physics is superconductivity where two electron fields can be combined to a scalar field with charge $2e$. Since we combine two spin half, the resulting spin can be 0 or 1. The first option amounts to defining

$$\phi = \psi_\uparrow(\vec{k})\psi_\downarrow(-\vec{k}) - \psi_\uparrow(-\vec{k})\psi_\downarrow(\vec{k})\,,$$

and write

$$e^{iS_{eff}[\phi]} = \int \mathcal{D}[\bar{\psi}, \psi]\delta[\phi - \psi_\uparrow(\vec{k})\psi_\downarrow(-\vec{k}) - \psi_\uparrow(-\vec{k})\psi_\downarrow(\vec{k})] e^{iS_0[\bar{\psi}, \psi]} \tag{75}$$

where $\delta[f(x)]$ is a *functional delta function*. The field ϕ is referred to as a Cooper pair field and describes two paired electrons with opposite spin and momenta. In a later section we shall come back to this problem and show how to carry out the integrals using a somewhat different approach.

3. Systematic integration of ψ_h – RG flow and phases of matter

The third way to define effective theories, is, I believe, the most profound one. Here the low energy d.o.f. are not qualitatively different from the high energy ones, but can be thought of as a coarse grained description. An example that you have met in your statistical mechanics course is the method of using "block spins ". To build on you previous knowledge, we now make a detour to explain the connection between QFT and statistical mechanics. A good starting point is the thermodynamic partition function,

$$Z(T) = \sum_n e^{-\beta E_n} = \mathrm{Tr}\, e^{-\beta H} \tag{76}$$

where $\beta = 1/kT$. In many statistical mechanics systems, the variables can be defined on a spatial lattice, the archetypical example being a lattice of classical spins. Denoting these with $\vec{s}_i$, the partition function is,

$$Z(T, \vec{m}) = \sum_{\{\vec{s}_i\}} e^{-\beta H(\vec{s}_i) - \vec{s}_i \cdot \vec{m}_i} \tag{77}$$

where we added a coupling to a set of auxillary variables $\vec{m}_i$, which can be used to calculate expectation values like $\langle \vec{s} \rangle$ and correlation functions like $\langle \vec{s}_i \vec{s}_j \rangle$, where $\langle \dots \rangle$ means the ground state expectation value.

You should now notice that (77) is very similar to the expression (42) for the QFT pathintegral. If you make the substitutions,

$$t \to -i\tau \quad ; \quad \mathcal{D}[\phi(x)] \to \sum_{\{\vec{s}_i\}} \quad ; \quad S_E \to H\,, \tag{78}$$

where S_E is the Euclidean action, and consider the trace of the evolution operator in imaginary time from $\tau = 0$ to $\tau = \beta$, you will get (77). Since taking the trace means that $q_i(\tau = 0) = q_i(\tau = \beta)$, so we must impose periodic boundary conditions when calculating the path integral. The outcome of all this is that:

Euclidean QFT in D-dim spacetime ≡ Classical stat mech in D-dim space

This equivalence has been very useful since it allows for techniques developed in statistical mechanics to be taken over to QFT and vice versa.

For the future, we record a related, but distinct, result concerning quantum statistical mechanics. I give you the QM version for a single particle with Hamiltonian $H(p, q)$ since by now the generalization to QFT should be immediate. The object of interest is the partition function,

$$Z(T) = \mathrm{Tr}\, e^{-\beta H} \tag{79}$$

and since we recognize $e^{-\beta H}$ as the imaginary time evolution operator from $q_i(\tau = 0) = q_i \tau = \beta$, we immediately get,

$$Z(T) = \int \mathcal{D}[q(t)] e^{-\int_0^\beta d\tau\, L(q,\tau)}\,, \tag{80}$$

so we conclude that

Euclidean QFT in D-dim spacetime with $\tau \in [0, \beta]$ ≡ Quant stat mech in d=(D-1)-dim space

Moving back to the equivalence between Euclidean QFT and classical statistical mechanics, we can now, as promised, borrow the very powerful techniques, of block spinning and renormalization group flow, that you have met in your course on statistical mechanics. Just as we discussed in the beginning of this section, the idea is to define low energy variables, which in the case of a spin system is basically the spin of a bigger chunk of the lattice. The next step is to integrate out the original spins to get an effective action in terms

of the new "coarse grained" spins. In this process you generate new interactions, and as you repeat the procedure you will hopefully reach a *fixed point of the RG flow*. In the next section we will see how this works in the archetypical case of superconductivity.

B. An example: the Renormalization Group approach to Fermi liquids

Although I assume basic knowledge about RG techniques, I will nevertheless provide you with a very short recap of the basic ideas and techniques. What might be new in this section is that I will work in momentum, rather than coordinate space.

1. The basic idea of the RG method

The goal of the RG method is to find an effective action, or Hamiltonian, that captures the low energy/low momentum properties of a theory. The technique is that of sequential elimination of high momentum degrees of freedom. Starting from a microscopic action $S = S_\Lambda$, the RG method generates a sequence S_{Λ_n} of (in general increasingly complicated) actions, where Λ_n is a cutoff and S_{Λ_n} describes the physics for momenta $p < \Lambda_n$. The first Λ can be thought of as a physical cutoff such as a lattice spacing in a crystal, or the magnetic length in strong magnetic field. Mathematically this elimination procedure is most easily described using path integrals, and the object to study is the partition function $Z[T, \mu, \dots]$, that depends on some number of control parameters such as the temperature T, the chemical potential μ *etc.*. Z can be used to calculate thermodynamic observables. To get correlation functions, or Greens functions in a quantum mechanical theory, one must also couple sources for the various fields. In the Euclidean formulation the correlation functions are directly related to retarded response functions, but by analytical continuation one can also, at least in simple cases, retrieve real time correlation functions. In the following we will suppress the dependence on external parameters and sources and write

$$Z = \int \mathcal{D}[\phi, \phi^\star]\, e^{-S[\phi]}\,. \tag{81}$$

For simplicity you can think of ϕ as a bosonic field that describes, for instance, the density in a gas or the spin density in some direction in a magnet. In general, there will be many fields, and some of them will be fermionic. We shall assume that the action is some local function of the field ϕ and its derivatives. As an (important) example we take

$$S = \int d^d x \, \left[\frac{1}{2}\phi^\star(-\nabla^2 + g_2)\phi + g_4|\phi|^4\right] = S_0 + S_{int} \tag{82}$$

which for for $d = 4$ could describe a cloud of cold atoms or a part of the Higgs sector of the standard model.

Next we fourier transform to momentum space and decompose the field as

$$\begin{aligned} \phi(\vec{x}) &= \sum_{0\le|\vec{p}|\le\Lambda} e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p}) = \sum_{0\le|\vec{p}|\le\Lambda_1} e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p}) + \sum_{\Lambda_1\le|\vec{p}|\le\Lambda} e^{i\vec{p}\cdot\vec{x}}\phi(\vec{p}) \\ &\equiv \phi_<(\vec{x}) + \phi_>(\vec{x}) \end{aligned} \tag{83}$$

The action becomes

$$\begin{aligned} S &= \sum_{0\le|\vec{p}|\le\Lambda_1} \frac{1}{2}(p^2+g_2)\,\phi^\star(\vec{p})\,\phi(\vec{p}) + \sum_{\Lambda_1\le|\vec{p}|\le\Lambda} \frac{1}{2}(p^2+g_2)\,\phi^\star(\vec{p})\,\phi(\vec{p}) + S_{int} \\ &= S_0[\phi_<] + S_0[\phi_>] + S_{int}[\phi_<,\phi_>] \end{aligned} \tag{84}$$

and we define the effective action S^{eff} at scale $\Lambda_1 = \Lambda/s$ by

$$e^{-S^{eff}[\phi_<]} = e^{S_0[\phi_<]} \int \mathcal{D}[\phi_>]\, e^{-S_0[\vec{p}_>]-S_{int}[\phi_<,\phi_>]} \tag{85}$$

We shall not go into the details for how S^{eff}, is calculated, but take for granted that this can be done, and just state that after the two rescalings,

$$\vec{p}_< \;\to\; \frac{1}{s}\,\vec{p}\,' \tag{86}$$

$$\phi_< \;\to\; \zeta\,\phi' \tag{87}$$

it can be written in the form

$$S^{eff} = \int d^dx\, \left[\phi^\star(-\nabla^2+g_2')\phi + g_4'|\phi|^4 + g_6|\phi|^6 + g_{22}|\nabla^2\phi|^2 + ...\right]. \tag{88}$$

where we skipped the primes on the new field variables. The parameter s in the scaling relation (86) is chosen so that the original cutoff Λ is restored, and the parameter ζ in (87) so that the kinetic term in S and S^{eff} are the same. (In a free field theory simple dimensional analysis gives $\zeta = s^{1-d/2}$, but for interacting field theories this is no longer true; the scaling dimension of the field becomes "anomalous".) This is necessary for making a meaningful comparison between the original coupling constants g_2 and g_4 and the new coupling constants g_2' and g_4'. By studying the "flow" of the coupling constants as we change the scale parameter s, we learn whether a particular interaction term becomes stronger or weaker as we study phenomena at lower and lower momenta. At first this might seem like an impossible endeavor since the dots in (88) denote an infinite number of terms of higher order in both ϕ and derivatives, which are generated when evaluating the functional integral in (85). What saves the day is that these operators[1] can be classified in three groups, *relevant, irrelevant, and marginal*, according to whether they increase, decrease, or remain the same under the *renormalization group transformation* which consists of the elimination of high momentum modes according to (85) and then performing the rescaling (86) and (87). In most cases there are only a few relevant and marginal operators, and these are the only ones that will survive at low momenta. Which class a particular operator belongs to often follows from dimensional analysis. In the action (82), the (mass) dimension of g_2 is that of ∇^2, *i.e.* $[g_2] = 2$ while $[g_4] = 4-d$ since $[\phi] = d/2-1$ and the action itself is dimensionless. Under the scale change (86), $g_2 \to g_2' = s^2 g_2$ and $g_4 \to g_4' = s^{4-d} g_4$, so

[1] Although we use path integrals, we shall refer to the terms in the action as "operators". In a Hamiltonian formalism, this is the natural language, and what corresponds to integrating out high momentum field components in the path integral is a reduction of the Hilbert space by eliminating the high momentum degrees of freedom.

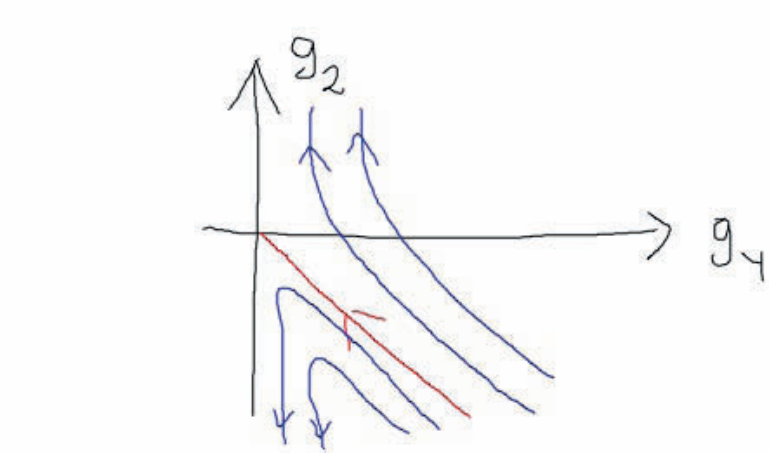

Figure 3 The RG flow generated by the equations (90). The Gaussian fixed point is at the origin

we expect g_2 to be relevant and g_4 relevant or irrelevant depending on whether we are below or above four dimensions.[2]

Mathematically, the flow of coupling constants are determined by a set of coupled first order differential equations of the type

$$s\frac{dg_i}{ds} = \frac{dg}{dt} = \beta_i(g_1, g_2, \dots) \tag{89}$$

where we introduced the parameter t by $s = e^t$. There is no universal method for calculating the beta functions, but in many cases one can get important information from various perturbative expansions. In our case, a one-loop calculation for $d = 4$ gives

$$\begin{aligned} \frac{dg_2}{dt} &= 2g_2 + ag_4 \\ \frac{dg_4}{dt} &= -bg_4^2 \end{aligned} \tag{90}$$

where a and b are positive constants. The resulting flow pattern in the (g_2, g_4) plane is shown schematically in Fig. 3 (make sure that you understand the qualitative features of this graph).

A *fixed point* is a set of values, here $(g_2^\star, g_4^\star)$, for the couplings which are unchanged under renormalizations, and from (89) we see that this occurs at zeros of the beta functions. The interesting fixed points are those at $s \to \infty$, since these determine the low energy theory.[3] A particularly important example is the *gaussian fixed point* where $g_2^\star = g_4^\star = 0$, as seen in the figure. In particle physics, this describes a free massless theory, and in statistical mechanics a critical point signalling a phase transition. A famous example of a non-Gaussian fixed point is the Wilson-Fisher fixed point which appears in the real version of the model (82) in three dimensions.

[2] The marginal case, $d = 4$ is more tricky and it takes a more careful analysis to find out whether the interaction is truly marginal, *marginally relevant* or *marginally irrelevant*

[3] In high energy physics it is also of interest to study "ultraviolet fixed points" corresponding to $1/s \to \infty$, which govern the short distance behavior of the theory.

2. Renormalization group approach to Fermi systems

To understand metals, one must understand the properties of an interacting electron gas/liquid in a background field of positive ions. An unreasonably good model is that of a free electron gas – unreasonably since the presence of strong Coulomb forces should naively make such a model meaningless. The reason for the success lies in the concept of a Landau liquid. Landau's idea was that the net effect of the interactions was to "dress" the electrons with a "cloud" of other electrons. The resulting *quasiparticles* have a mass that differ from the "bare" electrons, and the interact only weakly. Landu assumed that the "dressing" could be though of as an adiabatic process, so that the basic properties of the resulting "fermi liquid" were the same as that of the original electrons.

Although Landau's line of arguments was very suggestive, it cannot be true in general. We know that many metals at sufficiently low temperatures become superconductors which are very different from metals. Of particular importance for our discussion is that in superconductors there are no low energy quasiparticle excitations, which is related to that the Fermi surface is destroyed by interactions. Since one can argue that Landau's theory should be applicable at sufficiently low temperature, the existence of superconductors is a real problem.

The first go at this problem was to start from the free electron gas and take the electron-electron interaction into account by many-body perturbation theory. The first aim of this program, which was to derive Landau's theory from first principles, was rather successful, and it is described in many textbooks. Since the interaction is strong, it is not sufficient to calculate just a couple of terms, but one must perform (infinite) sums over many terms in the perturbation expansion - this makes the calculations cumbersome, and it is not always easy to keep track on what is included, what is neglected, and why that is so. The case of superconductivity is rather interesting. The first microscopic explanation - the BCS theory - was based on a rather simplified model for the electron-electron interaction, which, however, captured the essential physical mechanism, namely that of formation of "Cooper pairs" of two electrons. Only later was this theory derived using the perturbation theory machinery, and below we shall outline how it can be derived using QFT methods.

The second approach, which we shall discuss very briefly in this lecture, is based on Ken Wilson's formulation of the renormalization group (RG). We first briefly review the basic idea of Wilson's RG method, and then show how it can be used not only to get a more fundamental understanding of the Fermi liquid, but also to point to the possibility of a superconducting phase.

We now apply the methods from the last section to a system of interacting fermions, and for this we need the fermion path integrals that we discussed earlier. At finite chemical potential, there is a crucial qualitative difference between bosons and fermions in how the renormalization group transformations are defined. This difference is illustrated in Fig. 4 where we show how the cutoff changes if we do not perform any rescaling - this is the adequate way to illustrate which physical momenta are described by the effective theory as we follow the RG flow. The crucial point is that for fermions, the fixed point theory is not defined

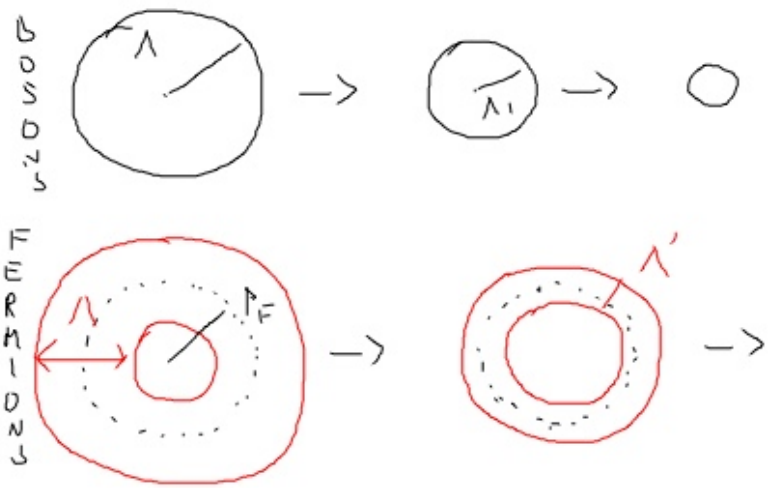

Figure 4 For bosons the RG transformation restricts to momenta to a smaller and smaller ball around $p = 0$ while for fermions the momenta are restricted to a smaller ands smaller *shell* around the Fermi momentum p_F.

by the values of the couplings as the sphere in momentum space shrinks to a point, but as the the values of the couplings on the Fermi sphere. Thus we do not have a set of fixed point coupling *constants*, but rather several fixed point coupling *functions* of the position on the Fermi surface.

3. The RG flow equations

To see a little bit more in detail how this comes about, we now study the scattering between two particles close to the Fermi surface, $1 + 2 \to 3 + 4$. We take two spatial dimensions to simplify the geometry (three spatial dimensions is discussed in detail in the article: R. Shankar, *Renormalization-group approach to interacting fermions*, Rev. Mod. Phys. 66, 1994). The Fermi surface is labeled by the polar angles $\theta_i \quad i = 1, \dots 4$, and an arbitrary two-momentum can be parametrized as $\vec{p} = (p_F + p)\vec{\Omega}$, where $\vec{\Omega} = (\cos\theta, \sin\theta)$. As in the previous section we now consider a four particle interaction, which in momentum space can be written as,

$$S_{int} = \int \prod_{i=1}^{4} d\omega_i \sum_{\vec{p}_i} \psi^\dagger_{\vec{p}_3} \psi^\dagger_{\vec{p}_4} \psi_{\vec{p}_2} \psi_{\vec{p}_1} \, f(\omega_i, k_i, \theta_i) \, \delta^2(\sum_i \vec{p}_i) \delta(\sum_i \omega_i) \, . \tag{91}$$

Here we distinguished between energy and momentum, used ψ instead of ϕ to indicate that we are dealing with fermions, and f instead of g_4 to anticipate the connection to Fermi liquid theory. The delta functions imposes energy and momentum conservation. Under the RG transformation, $k_i \to k_i/s$ and $\omega_i \to \omega_i/s$, while the angles θ_i remain unchanged. Thus we expect that any low momentum fixed point, corresponding

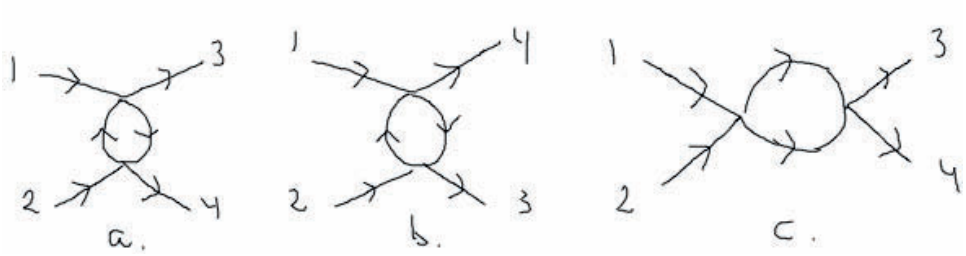

Figure 5 One loop contribution to electron-electron scattering

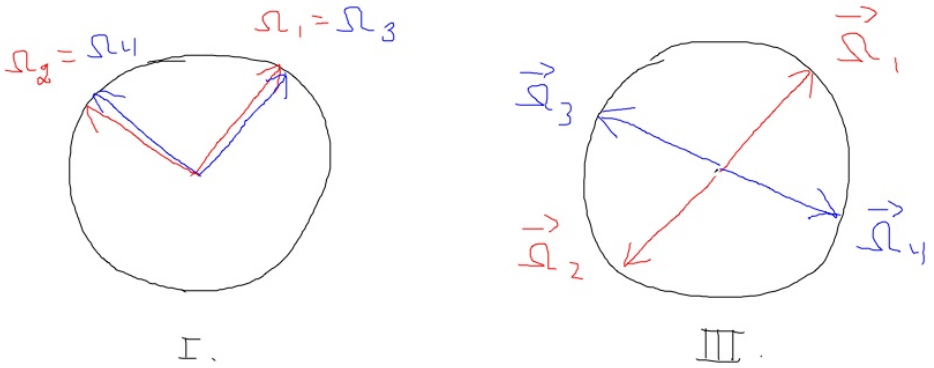

Figure 6 The direction of the momenta of particle scattering close to the Fermi surface. I and III refers to the labels in (92) and (95).

to $s \to \infty$ should be characterized not by a coupling constant $f^\star$, but by a coupling *function* $f^\star(\theta_i)$.

To get some handle on whether there are any such fixed point functions, and what properties they have, we must find the RG flow equations. The details of the perturbative calculations you find in the article by Shankar, but to understand the results, we must know a few things about how they are derived. The idea is to perform the integration in (85) using perturbation theory to one loop. We should thus evaluate the scattering graphs i figure 5. The important point is now that for all the momenta to lie close to the Fermi circle, there are only three possibilities for the vectors $\vec{\Omega}_i$, as illustrated in figure 6. The first two are,

$$\vec{\Omega}_1 = \vec{\Omega}_3 \quad \text{and} \quad \vec{\Omega}_2 = \vec{\Omega}_4 \qquad (I) \tag{92}$$

$$\vec{\Omega}_1 = \vec{\Omega}_4 \quad \text{and} \quad \vec{\Omega}_2 = \vec{\Omega}_3 \qquad (II) \tag{93}$$

which are identical up to an exchange of the final particles. This process is possible for arbitrary angles θ_1 and θ_2, and because of rotational invariance, we expect this scattering to be characterized by a fixed point function,

$$F(\theta_1 - \theta_2) = f(\theta_1, \theta_2, \theta_1, \theta_2) = -f(\theta_1, \theta_2, \theta_2, \theta_1) \tag{94}$$

where the sign comes from the particles being fermions. The third possibility is

$$\vec{\Omega}_1 = -\vec{\Omega}_2 \quad \text{and} \quad \vec{\Omega}_3 = -\vec{\Omega}_4 \qquad (III)\,, \tag{95}$$

that is scattering between particles at opposite positions on the Fermi circle. Here, the coupling function can depend only on the angle $\theta_1 - \theta_3$,

$$V(\theta_1 - \theta_3) = f(\theta_1, -\theta_1, \theta_3, -\theta_3)\,. \tag{96}$$

Here we just used geometric intuition to put in the constraints on the angles "by hand", but a careful evaluation of the diagrams in figure 5, taking the cutoff into account, will give the same result. Furthermore it will give the following RG flow equations for the functions $F(\theta)$ and $V(\theta)$,

$$\frac{dF(\cos\theta)}{dt} = 0 \tag{97}$$

$$\frac{dV(\cos\theta)}{dt} = -\frac{1}{4\pi}\int \frac{d\theta'}{2\pi} V(\theta - \theta')V(\theta')\,. \tag{98}$$

Noting that the RHS of (98) is a convolution, we can find a solution for the Fourier components V_L,

$$V_L(t) = \frac{V_L(0)}{1 + V_L(0)t/(4\pi)}\,, \tag{99}$$

where $V_L(0)$ are the starting values for the RG evolution (verify this solution!).

4. Physical interpretation

We are now finally in the position to to draw some conclusions about the low energy theory. First, assume that allnthe $V_L(0)s$ are positive, as would be the case for a reasonably behaved repulsive potential, then $V(\theta)$ will renormalize to zero,[4] while the function $F(\theta)$ remains marginal and characterizes the fixed point theory which can be identified as a Fermi liquid!

If instead at least one of the $V_L(0)$ is negative, as would be the case for any attractive interaction, the RG flow will hit a singularity for some t. At this point the coupling constant, V_L grows out of control, and the perturbative treatment can no longer be trusted. From the details of the calculations, one can, however, see that the contribution to the renormalization of $V(\theta)$ comes from the "BCS" diagram in figure 5c. We can thus interpret this singularity as the emergence of a pole in the L^{th} partial wave of the particle - particle scattering amplitude, corresponding to the formation of a "Cooper pair" with angular momentum L. In section III.D we shall use the insights about the instability towards Cooper pair formation to derive an effective action for a BCS superconductor.

We have glossed over several important technical points such as how to properly include spin and and the constraints imposed by the Pauli principle. It also takes some work and care to establish the flow equations (97) and (98) from perturbation theory, and to make sure that the resulting putative Fermi liquid, actually coincides with the phenomenological theory formulated by Landau.

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[4] This is not strictly true. At sufficiently low temperatures, an additional contribution, that was neglected in (98) becomes important, and the system is driven towards superconductivity. This is the RG manifestation of the so called Kohn-Luttinger effect.

C. Symmetries and Symmetry Breaking

We now turn to a more detailed discussion of symmetries, and will also introduce the important concept of *symmetry breaking*. In relativistic QFTs, the important symmetries are Poincare symmetry, which is composed of translations in space and time, but also general Lorentz transformations. In addition, we have the discrete symmetries of parity (P), time reversal (T), which are related to space-time, but also particle-antiparticle conjugation (C). Of these discrete symmetries, both P, and CP can be violated in weak processes. There is a fundamental theorem in relativistic QFT, which I will not attempt to prove, which states that the combined symmetry CPT, must remain unbroken in any Lorentz invariant local QFT. An important implication of this symmetry is that particles and antiparticles must have the same mass. Since CP is known to be violated, we can thus conclude that T must also be violated in some processes. Charge conjugation is not acting in space-time and is thus an "internal symmetry", and there are also continuous internal symmetries, such like isospin symmetry. Such a symmetry can, for instance, change a proton into a neutron.

There is another type of "symmetries", called gauge symmetries. The quotation mark is since these are not a symmetry in the ordinary sense, where a symmetry transformation changes the state of the physical system, like a rotation, a translation, or an isospin rotation. We will discuss these gauge symmetries in a later lecture.

In statistical mechanics you have already encountered the concept of *symmetry breaking*, and *order parameter*. Typically the low temperature state is characterized by a non-vanishing expectation value of the order parameter, and the archetypical example is a ferromagnet below the Curie temperature. At higher energies the moments become disordered and the magnetisation vanish.

Much of the above logic can be taken over to Euclidean QFT at zero temperature, which as we learned, corresponds to a classical statistical system if we identify the Euclidean action with the Hamiltonian of the statistical system. The thermal fluctuations in the classical system correspond to the quantum fluctuations in the Euclidean QFT. This allows you to study *quantum phase transitions*, which occur at zero temperature when you vary some control parameter such as density, strain, magnetic field, etc. A typical example is the loss of superconductivity when an external magnetic field exceeds a certain critical value.

D. An example: The BCS theory of superconductivity

We now return to the example in Sect. III.A.2 that illustrates several of the concepts we discussed in this lecture – introduction of auxiliary fields, the derivation of an effective field theory for an order parameter, the effective action for external fields, and spontaneous symmetry breaking.

The starting point is the observation in the previous section that there is a potential instability of the Fermi surface due to strong interactions between electrons with opposite momenta and spin. This "Cooper instability" makes it interesting to consider the possibility of having Cooper pairs, i.e. bound states of such

pairs of electrons. We now explore this possibility in a QFT context.

The aim is thus to calculate the effective action for a scalar field, $\phi = \psi_\downarrow \psi_\uparrow$ that describes a Cooper pair. Here we use an oversimplified description of a Cooper pair. Real electrons cannot be at the same point in space due to the strong Coulomb repulsion, so the pairing is really in momentum space as we described earlier. Disregarding the finite size of the pairs is for pedagogical purposes. With this we proceed to calculate the effective action heuristically given in (75).

We shall use the action,

$$S_E[\bar{\psi},\psi] = \int_0^\beta d\tau \int d^d r \sum_{\sigma=\uparrow,\downarrow} \bar{\psi}_\sigma \left(\partial_\tau + ieA_0 + \frac{1}{2m}(-i\nabla - e\mathbf{A})^2 - \mu \right) \psi_\sigma - g\bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \,, \tag{100}$$

which describes spinful electrons at chemical potential μ coupled to the electromagnetic field A_μ, and interacting via a pointlike interaction with strenght g. I wrote it in imaginary time τ to indicate how to do the calculation at finite temperature, $1/\beta$ but in the following I will ignore the temperature. To evaluate the functional integral, which is not Gaussian, we shall use the following trick due to Stratonovich and Hubbard,

$$e^{g\int d\tau \int d^d r\, \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow} = \int \mathcal{D}(\bar{\phi},\phi)\, e^{-\int d\tau \int d^d r \left[\frac{1}{g}|\phi|^2 - \phi^\star \psi_\downarrow \psi_\uparrow + \phi \bar{\psi}_\uparrow \bar{\psi}_\downarrow \right]} \,. \tag{101}$$

With this, we can extract the effective action for the Cooper pair field ϕ from the Gaussian integral

$$e^{iS_{eff}[\phi]} = \int \mathcal{D}(\bar{\psi},\psi) e^{-\int d\tau \int d^d r \left[S_0 - \phi^\star \psi_\downarrow \psi_\uparrow + \phi \bar{\psi}_\uparrow \bar{\psi}_\downarrow \right]} \,. \tag{102}$$

where S_0 is the quadratic part of the fermionic action. The effective background action for the electromagnetic field can then be calculated from,

$$Z[A] = \int \mathcal{D}(\bar{\phi},\phi^\star) e^{-S_{eff}[\phi]} \,. \tag{103}$$

To do the Gaussian integral in (102) we introduce the Nambu spinor Ψ,

$$\Psi = \begin{pmatrix} \bar{\psi}_\downarrow \\ \psi_\uparrow \end{pmatrix}, \quad \Psi^\star = \begin{pmatrix} \psi_\downarrow & \bar{\psi}_\uparrow \end{pmatrix}, \tag{104}$$

and rewrite the fermionic part of the action as

$$S_f = \bar{\Psi} \mathcal{G} \Psi \tag{105}$$

with

$$\mathcal{G}^{-1} = \begin{pmatrix} G_p^{-1} & \phi \\ \phi^\star & G_h^{-1} \end{pmatrix}. \tag{106}$$

where p and h stand for particle and hole excitations respectively, and where

$$G_{(p)}^{-1} = -\partial_\tau - ie\phi - \frac{1}{2m}(-i\nabla - e\mathbf{A})^2 + \mu \qquad ; \qquad G_{(h)}^{-1} = -\partial_\tau + ie\phi + \frac{1}{2m}(+i\nabla - e\mathbf{A})^2 - \mu$$

. With this notation, you can (do it!!) evaluate the Gaussian integral in (102) to get

$$e^{iS_{eff}[\phi]} = e^{-\int d\tau \int d^d r \frac{1}{g}|\phi|^2 + \ln\det \mathcal{G}^{-1}} \,. \tag{107}$$

Although simple looking, this expression is rather complicated due to the determinant of the complicated kernel $\mathcal{G}^{-1}$. This is, however, a technical problem, and in the spirit of effective actions, one evaluates it in powers of ϕ and a low number of derivatives of ϕ. The answer for static configurations and to lowest non-trivial order is the celebrated Landau-Ginzburg functional,[5]

$$S_{LG} = \beta \int d^d r \, \frac{r(T)}{2}|\phi|^2 + \frac{c}{2}\phi^\star(-i\vec{\nabla} - e\vec{A})^2\phi + u|\phi|^4 \,. \tag{108}$$

where $u > 0$ and $r(T) > 0$ for $T > T_c$, and $r(T) < 0$ for $T < T_c$, where the critical temperature, T_c can be calculated. First we find the homogeneous ground state in the absence of electromagnetism, by warying with respct to the density $\rho_0 = |\phi|^2$,

$$\rho_0 = \frac{r(T)}{u} \quad \text{for } T < T_c \qquad ; \qquad \rho_0 = 0 \quad \text{for } T > T_c \,. \tag{109}$$

This means that for $t < T_c$ there is a non-vanishing density of Cooper pairs. What does this tell us about the order parameter field ϕ? For this recall that ϕ is complex so we can write it as $\phi = \sqrt{\rho}e^{i\theta}$. So for ϕ to take a constant value in the ground state, it is not enough to have a constant density, we must also have a constant phase. To understand if this is the case, we write the second term in (108) for constant $\rho = \rho_0$, to get

$$S_{LG} = \text{const.} + \int d^d r, \frac{c\rho_0}{2}(\vec{\nabla}\theta - e\vec{A})^2 \,, \tag{110}$$

so absent $\vec{A}$ the minimum is at constant θ which means that below T_c we have $\langle\phi\rangle \neq 0$ in the ground state, where $\langle..\rangle$ denotes the vaccum expectation value. This is an archetypicical example of a spontaneous breaking of a gauge symmetry, that will be discussed later.

From (110), we can also directly find the effective response action using (103). A simpler way is to note that the integration over ρ just fixes it to ρ_0, and that by the gauge transformation $\vec{A} \to \vec{A} - \vec{\nabla}\theta$, to the so called London gauge, the functional integral becomes independent of $\vec{A}$, and we are left with

$$S_{res}[\vec{A}] = \frac{c\rho_0 e^2}{2}\vec{A}^2 \,. \tag{111}$$

Combining this with the energy of the magnetic field, we get the *effective potential*,

$$V_{eff} = \frac{1}{2}B^2 + \frac{c\rho_0 e^2}{2}\vec{A}^2 \,, \tag{112}$$

and solving the corresponding differential equation for $\vec{A}$ close to an interface between a superconductor and vacuum (or an insulator) one finds that the magnetic field only penetrate the superconductor in a thin layer

[5] I have used a notation that is essentially that in the textbook *Condensed Matter Field Theory* by Altland and Simons which you should consult if you want to fill in the details and discover where I have (over)simplified.

$\sim \lambda_L$, where the London penetration length λ_L is given by $\lambda_L^2 = \frac{c\rho_0 e^2}{2}$. This is the Meißner effect, which is an important characteristic of a superconductor.

You might wonder if it is possible to have a finite density of Cooper pairs without having *phase coherence*, i.e. a constant θ. In the so-called high T_c superconductors that were discovered in the late 1980s, it is believed that in a part of the phase diagram, this can be the case.