Applied Analysis in the UK

Lucia Scardia

Heriot-Watt University

LMS Prospects, 10 September 2020

Research Themes in Analysis

- functional analysis
- measure theory
- probability
- PDEs
- dynamical systems
- numerical analysis
- calculus of variations
- optimal transport theory
- inverse problems

Calculus of variations

Calculus of variations: Infinite-dimensional optimisation

$$E(u) = \int_\Omega f(\nabla u(x))\, dx, \quad u : \Omega \to \mathbb{R}^3$$

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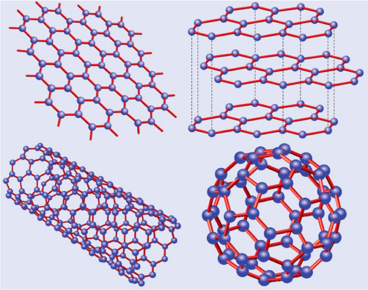
Many patterns in nature can be modelled as energy minimisers



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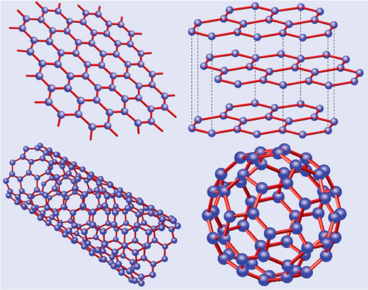

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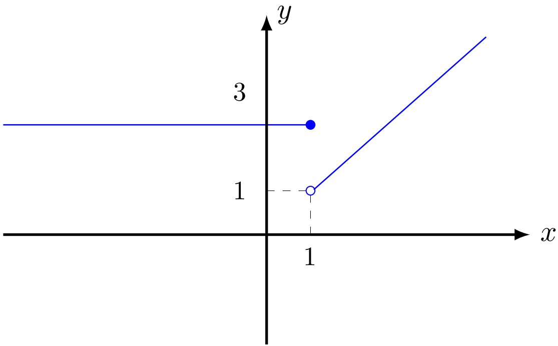

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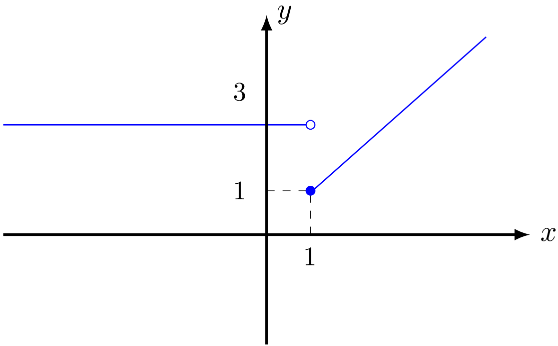

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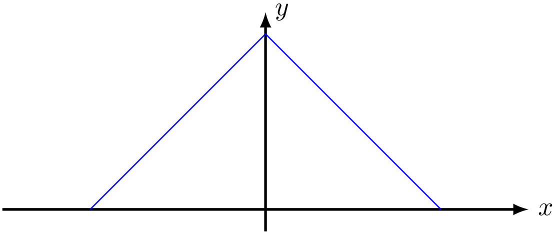

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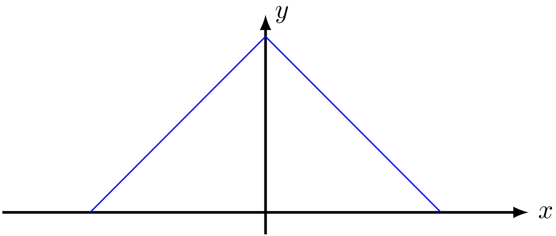

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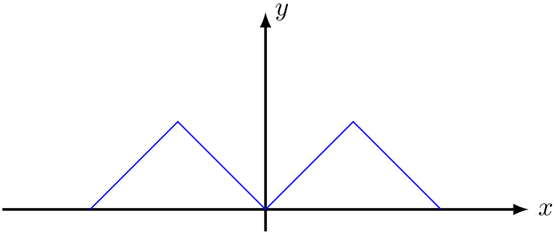

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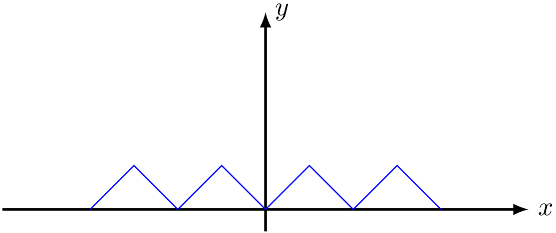

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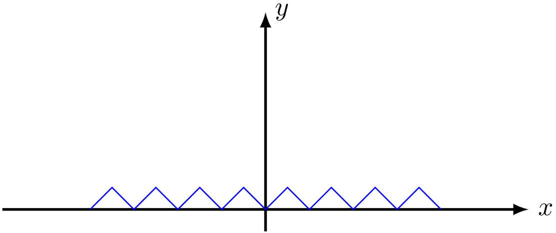

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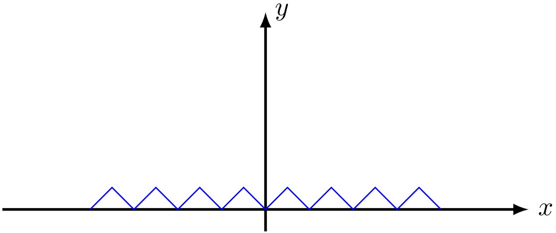


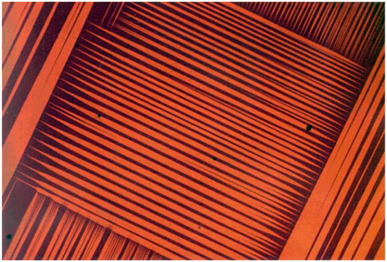
- To make the second term 0 we need $u = 0$
- E can be arbitrarily close to 0 but you can never have $E(u) = 0$.

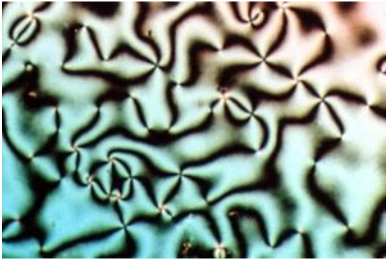
Calculus of variations

Alloys and liquid crystals

$$E(u) = \int_\Omega f(\nabla u(x))\, dx, \quad u : \Omega \to \mathbb{R}^3$$

Existence of minimisers under physical assumptions



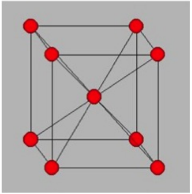


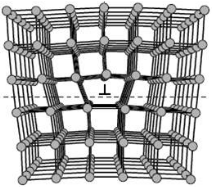
R.D. James: Materials from mathematics, Bull. Amer. Math. Soc. 56 (2019)

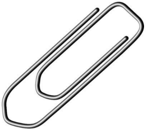
Interacting particle systems

$$E_n(x_1, \dots, x_n) = \frac{1}{n^2} \sum_{i,j=1}^{n} W(x_i - x_j)$$

- $x_i \in \mathbb{R}^3$ positions of n particles   e.g., atoms, charges, defects in metals, people, animals, cars, etc.
- W interaction potential   e.g. attractive or repulsive (or both), regular or singular, radial or anisotropic, etc.







Goal: Find energy-minimising configurations (Thomson problem)

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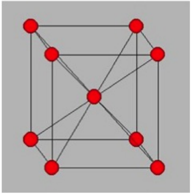

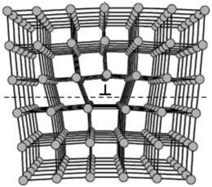

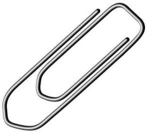

Problem: too many particles, explicit minimisation impossible!

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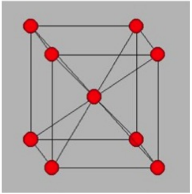
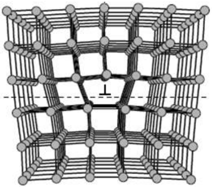
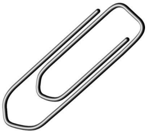






Idea: Derive a 'macroscopic' energy as $n \to \infty$ and minimise that!

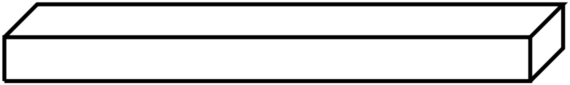
Interacting particle systems

$$E(\rho) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} W(x-y)\,\rho(x)\rho(y)dxdy$$

- ρ density of particles   e.g., atoms, charges, defects in metals, people, animals, cars, etc.
- E functional of continuum variable ρ less degree of freedoms, calculus of variations framework

Energy-minimising patterns are easier to compute and are a good approximation of minimisers of E_n for large n

Homogenisation: Simplest example



$$-u''(x) = g(x), \quad x \in (-1, 1)$$

Solution u: temperature distribution in a homogeneous bar with heat source g

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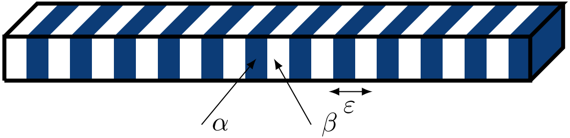

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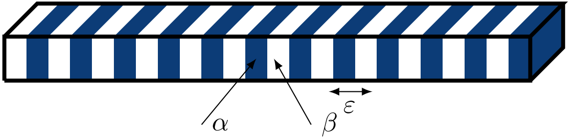

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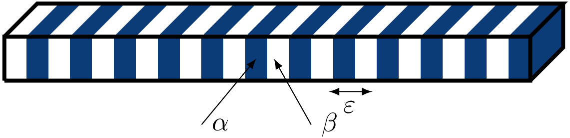

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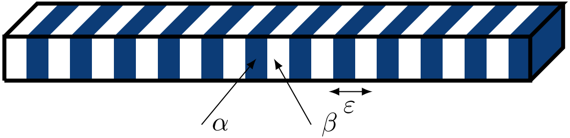

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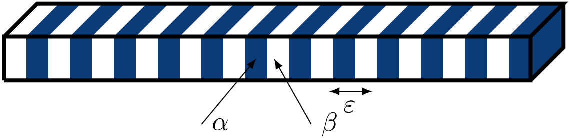

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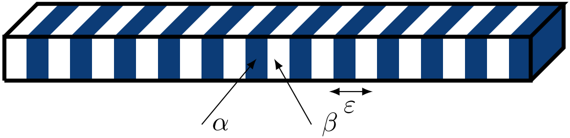

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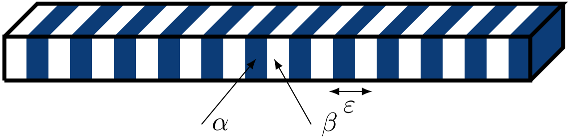

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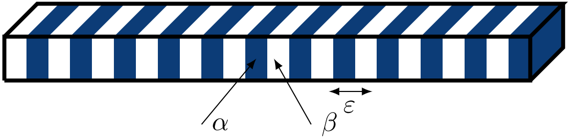

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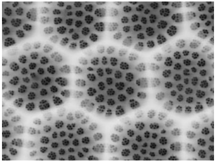
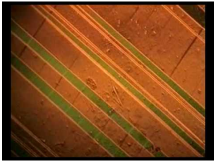
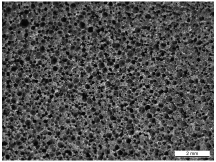
$$a_{\text{hom}} = \frac{1}{\left(\overline{\frac{1}{a}}\right)} = \left(\frac{1}{2}\int_{-1}^{1}\frac{1}{a(x)}\,dx\right)^{-1} = \frac{2\alpha\beta}{\alpha+\beta}$$

Homogenisation (deterministic and stochastic)

$$-\mathrm{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u(x)\right) = g(x) \quad \text{in } \Omega$$

- $u : \Omega \to \mathbb{R}^n$ e.g. temperature, deformation, displacement;
- a e.g. conductivity, elastic moduli;
- g e.g. heat source, applied force;

- Hierarchical, periodic, random microstructure at scale ε
- Highly oscillating at scale ε but ‘macroscopically’ homogeneous!

Optimal transport theory

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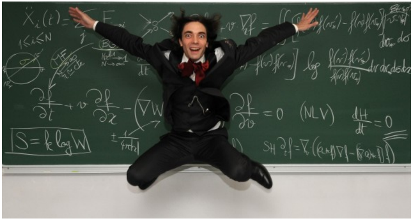
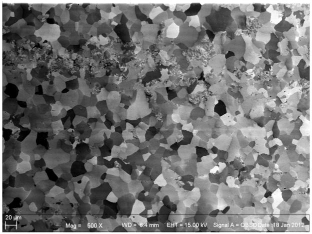
- 2003: Fields Medalist Cédric Villani publishes textbook on OT.

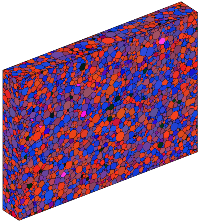


- July 2018: Alessio Figalli wins a Fields Medal for his work in OT.

Optimal transport theory

Applications: weather modelling, steel industry, machine learning, image processing, economics, materials science, etc.





Optimal transport theory

Image processing - Maxwell Institute PhD students

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