PhD Research in Numerical Analysis

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September 11, 2020

Introduction

NA is concerned with the **development** and **analysis** of algorithms for **approximating** solutions to problems, such as

- Integration: $\int_D f(\mathbf{x})\,d\mathbf{x}$,
- Interpolation / polynomial approximation: $p(\mathbf{x}) \approx f(\mathbf{x})$
- ODEs: $\frac{d\mathbf{u}}{dt} = \mathbf{f}(t, \mathbf{u}(t))$
- PDEs: $-\nabla^2 u(\mathbf{x}) = f(\mathbf{x})$
- Linear systems: $A\mathbf{v} = \mathbf{b}$
- Optimisation: $\min_{\mathbf{x}\in S} f(\mathbf{x})$
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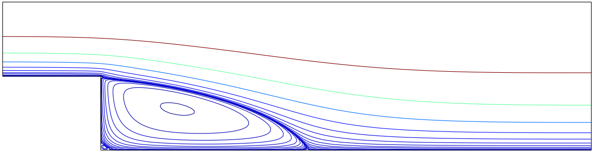
Modern research is driven by: **complex real-world applications**, evolving computer architectures and **increasing computing power**.

PDE Example: Navier–Stokes Equations

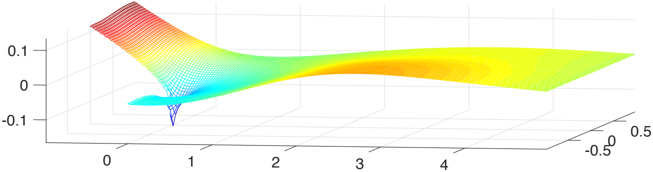
$$-\nu\nabla^2\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p \quad = \quad \mathbf{f}, \qquad \nabla\cdot\mathbf{u} = 0.$$

We can approximate **u** (the velocity) and p (the fluid pressure) using e.g. **finite element methods** (FEMs).

Streamlines: selected



Pressure field



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- Design a method to estimate e '**a posteriori**'.

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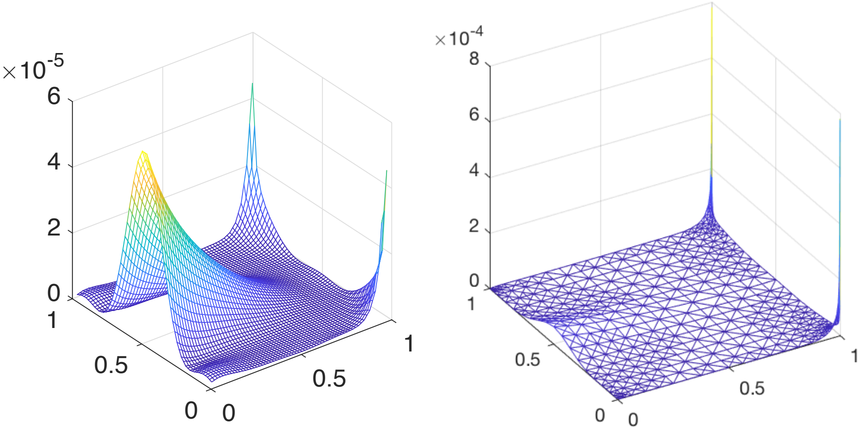
- Compute a **sequence** of approximations $u_1, u_2, \ldots$ until

$$\eta \leq TOL.$$

Non-adaptive vs. Adaptive FEM Approximation



Active Areas of NA Research in UK*

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- **Optimisation:** Fundamental theory, inverse problems, imaging applications and machine learning, compressed sensing, ...

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Project 1: Methodological (PDEs)

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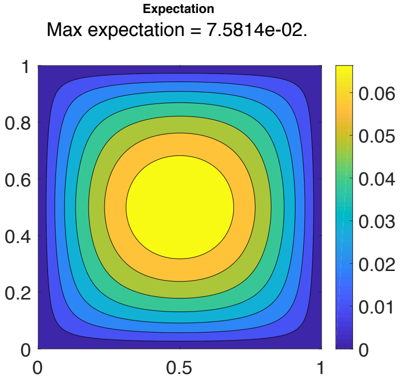
$$-\nabla \cdot \left(\underbrace{a(\mathbf{x}, \mathbf{y})}_{\text{coefficient}} \nabla u(\mathbf{x}, \mathbf{y}) \right) = f(\mathbf{x})$$

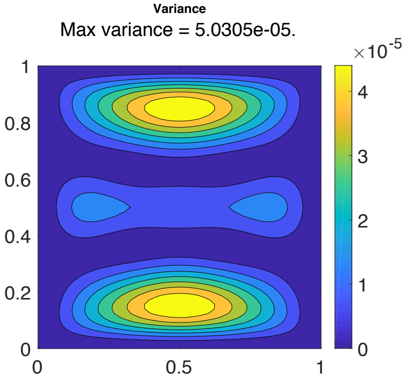
where the coefficient is a **parameter-dependent** function

$$a(\mathbf{x}, \mathbf{y}) = a_0(\mathbf{x}) + \sum_{m=1}^{\infty} a_m(\mathbf{x}) \underbrace{y_m .}_{\text{parameter}}$$

Numerical Results: Test Problem

Estimated **mean** (left) and **variance** (right) of $u(\mathbf{x}, \mathbf{y})$

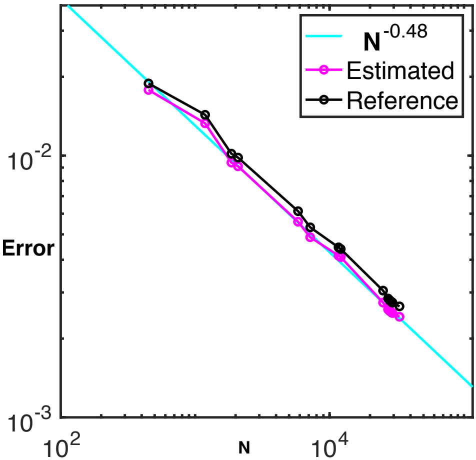




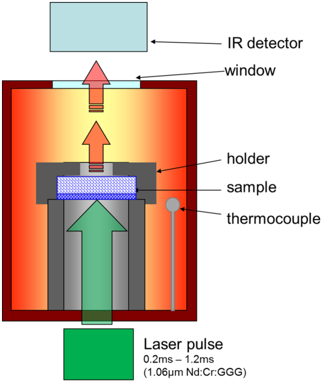
Estimated Error & Convergence Rate

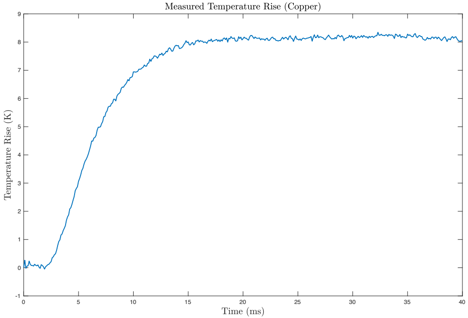
N:= dimension of approximation space.

Optimal convergence rate: $N^{-1/2}$ (for this test problem).



The Challenge: Approximate thermal conductivity λ of materials using data from laser flash experiments & estimate associated uncertainty.





Project 2: Application-Focused (iCASE PhD with NPL)

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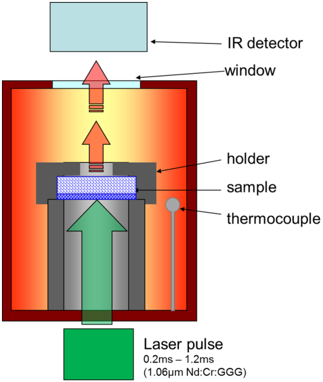
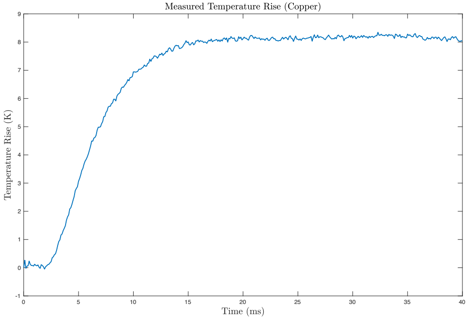
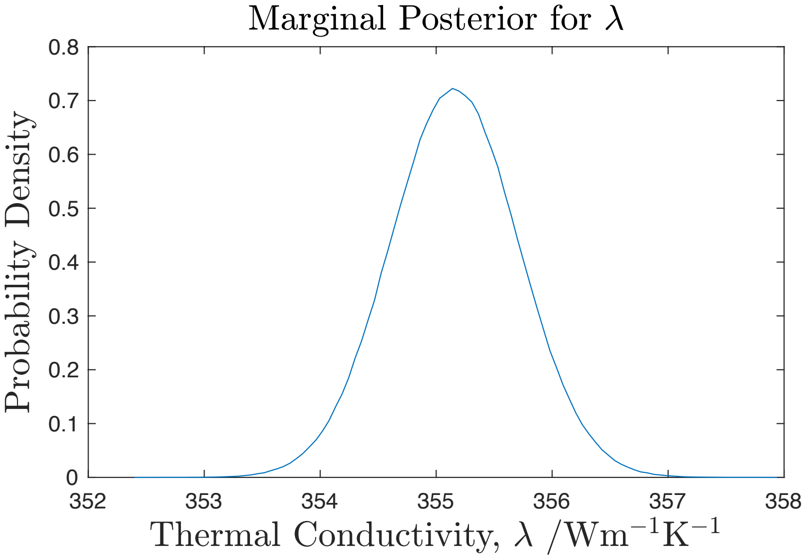
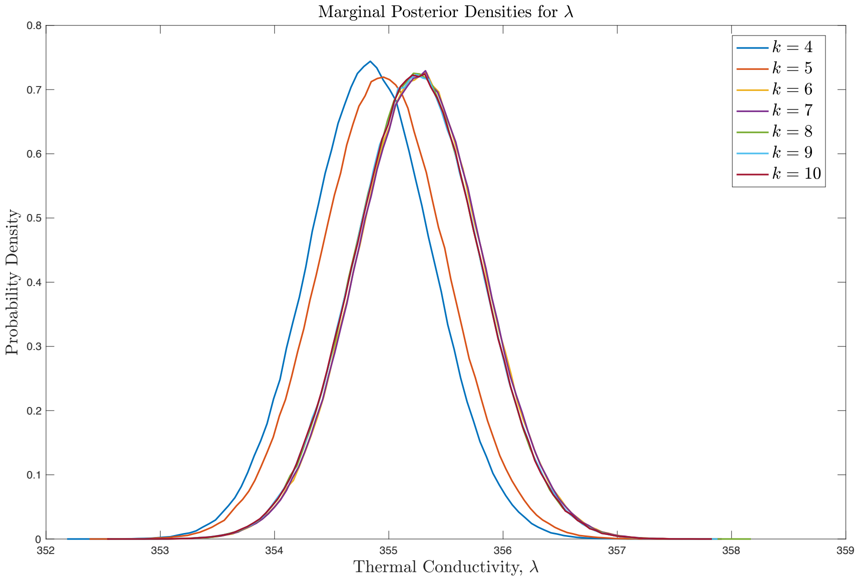




Approach: Formulate as a **Bayesian inverse problem** and approximate the conditional pdf of λ (given the data).


Marginal Posterior for λ
Probability Density
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
352
353
354
355
356
357
358
Thermal Conductivity, λ /$\mathrm{Wm^{-1}K^{-1}}$

Convergence (as polynomial degree $k \to \infty$)



Project 3: Methodological (Linear Algebra)

Challenge: Design adaptive iterative methods for solving **massive** linear systems $A\mathbf{v} = \mathbf{b}$ where

$$A = \underbrace{G_1}_{m \times m} \otimes \underbrace{K_1}_{n \times n} + G_1 \otimes K_2 + \ldots + G_M \otimes K_M,$$

and $\otimes :=$ the matrix 'Kronecker product'.

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Equivalently, approximate the **matrix** V satisfying the matrix equation

$$K_1 V G_1^\top + K_2 V G_2^\top + \ldots + K_M V G_M^\top = B$$

where $\mathbf{v} = \text{vec}(V)$ and $\mathbf{b} = \text{vec}(B)$.

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- **Applications:** Dynamical systems, high-dimensional & stochastic PDEs.

Things to think about before applying

- Try and **narrow down the area** (in NA) you want to work in
- Do you want to prove theorems, do computations, or both?
- Do you want to do **core methodological** research, or something **application-driven**?
- University webpages can be hard to navigate - look at **research** pages.
- **Look beyond CDTs** - not every institution has one.
- Attend (virtual?) **Open Days**.